Chaos, Solitons and Fractals 182 (2024) 114764

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Exploring social networks through stochastic multilayer graph modeling

Mohammad Mehdi Daliri Khomami [a], Mohammad Reza Meybodi [a,*], Alireza Rezvanian [b]

[a] Department of Computer Engineering, Amirkabir University of Technology, Tehran, Iran
[b] Department of Computer Engineering, University of Science and Culture, Tehran, Iran

ARTICLE INFO

Keywords:
Multilayer social networks
Cellular Goore Game
Stochastic multilayer graphs
Measurements

ABSTRACT

Several graph models are available today to model online social networks. These graph models are used to analyze the structural properties of the online social network, such as detecting communities, finding the influential spreader and predicting the behavior of the network. However, these models are based on deterministic single-layer graphs that may not be appropriate when online users use multiple social networks at the same time and social networks provide specific services. Moreover, because of the unknown and dynamic nature related to the behaviors and activities of online users, as well as structural and behavioral parameters, which may vary over time, stochastic multi-layer models could be applied to better capture and represent this phenomenon, as well as the dynamic nature of social networks. For example, in recommender systems, users' interests are unknown parameters and vary over time. Therefore, stochastic multilayer graph modeling can be used to develop recommender systems by considering different layers for different types of interests or preferences. In this paper, we propose a stochastic multilayer graph in which the edges are associated with random variables as a potential graph model for the analysis of online social networks. Therefore, after redefine some network measures related to stochastic multilayer graphs, we propose a Cellular Goore Game (CGG) based algorithm to computes the redefine network measures. A CGG-based algorithm computes defined network measures by learning automata from the edges of stochastic multilayer graphs. The experimental results show that the new CGG-based algorithm requires fewer samples from the edges of stochastic multilayer graphs than the standard sampling method in network measures calculation. Furthermore, the obtained results demonstrate that, from an evaluation perspective, the CGG-based algorithm provides superior results in terms of Kolmogorov-Smirnov (KS-test), Pearson Correlation Coefficient (PCC), Normalized Root Mean Square Error (NRMSE) and Kullback–Leibler divergence (KL-test).

1. Introduction

Social networks such as Facebook, Twitter, and LinkedIn have witnessed exponential growth among social media users. Social networking sites are increasingly popular as communication tools. The past few decades have seen a significant amount of research conducted in this particular context. However, many researchers assume the social network structure under study is single-layer. Even, several models have been developed to demonstrate the main characteristics of real networks, such as the Erdős-Rényi model [1], Watts-Strogatz model [2], Barabási-Albert [3] and Kroneker [4]. These models are presented to reveal the main characteristic of online social networks such as small-world [2], scale-free [3], community structure [5], heavy-tailed degree distribution [6] and small diameter. Single-layer network models, which assume that relationships between individuals are either present or absent, have been criticized for their inability to capture the complexity of real-world social interactions. One example of such a criticism comes from the field of sociology and concerns the concept of multiplexity in social networks. Multiplexity refers to the idea that social ties between users can involve multiple dimensions, such as emotional closeness, shared interests, and geographic proximity. Single-layer network models, however, only consider one type of relationship between individuals, such as friendship or coworker status. Research has shown that multiplexity has important implications for social influence, information diffusion, and collective action. For instance, studies have found that people are more likely to adopt new behaviors or beliefs if they receive consistent messages from multiple sources within their social network, rather than just one. Therefore, Single-layer network models may overlook important aspects of social relationships that could have significant effects on individual behavior and group

* Corresponding author.
E-mail addresses: M.daliri@aut.ac.ir (M.M.D. Khomami), mmeybodi@aut.ac.ir (M.R. Meybodi), rezvanian@usc.ac.ir (A. Rezvanian).

https://doi.org/10.1016/j.chaos.2024.114764
Received 1 April 2023; Received in revised form 23 June 2023; Accepted 18 March 2024
Available online 27 March 2024
0960-0779/© 2024 Elsevier Ltd. All rights reserved.

dynamics. From another aspect, due to the fact that online users often use several social networks simultaneously and since these networks are specialized in providing services, modeling them as a single-layer may be inappropriate. In single-layer representations, each node is typically regarded as a user of the network, and each interaction is a connection (or edge). Furthermore, it is clear that treating all network changes as a single-layer network may cause a significant constraint, as well as result in the loss of some critical information if all changes are treated as single-layer network. As an example, consider a social network, which is described as a collection of individuals (or groups of individuals) with a pattern of interaction between them [14,15]. It seems natural to presume that all relationships or connections between users take place at the same level. However, in reality, the distance between users is substantial, and relationships may occur at different levels of relationship. Also, certain parameters of network structures, such as behavioral characteristics and structural characteristics, are subject to time variations because of the dynamic characteristics of many decision-making problems in real-world domains. In order to come closer to an understanding of the characteristics of these networks, stochastic graphs are presented as a model [7]. In [7], it was found that stochastic graphs in which the edges are random variables are more suitable for analyzing social networks. In addition, they provided a comprehensive definition of network characteristics associated with stochastic graphs. Based on the introduction of stochastic graphs for social networks analysis, the concept of trust and link prediction in stochastic networks has been studied in [8,9]. Among the key factors affecting trust in stochastic networks, it has been shown that the length of trust paths is one of the most important influences on trust [8]. Also, they stated the quality of trust is affected by the length of trust paths. A learning-based method is used by the authors in order to aggregate the trust values derived from multiple paths together in order to create a single trust value. They also noted that trust inferences are much more likely to be accurate if trust values are evaluated based on all paths. Nevertheless, in large social networks it can be very time-consuming to compute all the paths. Therefore, they have developed a dynamic algorithm to infer trust networks using distributed learning automata. In [9] the link prediction problem has been studied in stochastic graphs. With a dynamic and uncertain concept of activities in social networks, research has shown that over time, the structure of networks can also change. A link prediction algorithm based on learning automata has been designed for stochastic social networks by taking advantage of this concept and has been developed by the company using the potential for future relationship formation.

Considering the above points regarding user activity and stochastic modeling, all studies assume that the network under study is single-layer. However, recently, a new phenomenon has emerged in social networks that enables the analysis of user behavior across multiple levels of activities, referred to as multilayer networks. Multilayer social networks are a potential solution to studies effectiveness in understanding user behavior because they allow researchers to analyze user interactions across multiple platforms or layers of social networks. As users increasingly use multiple social media platforms, it becomes important for researchers to understand how these different platforms interact with each other and how user behavior on one platform can affect their behavior on another platform. By studying multilayer social networks, researchers can gain a more comprehensive understanding of how people use social media and how information flows through these networks. They can also identify patterns of behavior that might not be evident if only one social network is studied in isolation. For example, in terms of community detection, in a multi-layer graph, nodes can belong to multiple communities across different layers. This means that the analysis of community structure can take into account the relationships between nodes across multiple layers, providing a more nuanced understanding of the system being studied. Also, multi-layer graphs allow for the consideration of different types of relationships between nodes, such as co-authorship, citation, or friendship, each of which may have their own distinct community structure.

Since users' behavior in social networks with multiple layers, such as their use of different social networks, their friendship behaviors, and their commenting behavior, vary over time with unknown probabilities. Using deterministic multilayer graphs to model social networks is not sufficient to reflect what users are doing in different social networks as time goes on. Even representation of multilayer networks with different weighted network in each layer can only provide a snapshot of the real network. Furthermore, the topology and structure of multilayer social networks depend on the number of user connections between layers and their connections with other networks (layers). The use of stochastic multilayer graphs with weights connected to edges in various layers as random variables seems a promising candidate for modeling real-world network applications. The selection of a stochastic multilayer graph as the graph model will allow all the characteristics, measures, and concepts associated with the multilayer graph to be interpreted as stochastic features, such as degree, strength, clustering coefficient, cover, clique, and spanning tree. By modeling online social networks using stochastic multilayer graphs, we defined influential nodes based on a set of vertex cover nodes associated with viral marketing and a random variable associated with each node. It is possible to apply stochastic vertex cover to the study of influence maximization in the application of viral marketing to stochastic multilayer social networks as a graph model. In [10] a social choice procedure, social agents are modeled as Nash games. Through an undirected graph-based social communication network, the agents communicate and their opinions are governed by dynamic rules. The criteria used by the agents in this game describe a trade-off between self-consistent and manipulative behavior. It has been shown that the best response strategies result in dynamic rules for their actions.

Stochastic multilayer social network provides a more precise and intricate representation of human behavior. Stochastic multilayer graphs are more suitable for modeling the unpredictable and time-varying nature of social network activities because they allow for the representation of the dynamic interactions between individuals in a social network. Social networks are complex systems that are characterized by their high degree of interconnectedness, dynamic, unpredictable and time vary. In order to accurately model these characteristics, it is necessary to use a framework that can capture these properties of the system as it evolves over time. Stochastic multilayer graphs provide such a framework by allowing for the representation of multiple layers of connectivity between individuals in a social network. Each layer represents a different aspect of the network, such as the frequency and duration of interactions, the strength of ties between individuals, or the content of communication. By modeling these different aspects of the network separately, stochastic multilayer graphs can capture the heterogeneity and complexity of social network activities. Furthermore, stochastic multilayer graphs are capable of incorporating unpredictability and uncertainty into their models. This is important because social network activities are inherently unpredictable and has time varying nature. In real applications such as influence maximization and budget allocation are critically important in competitive environments. However, if we fail to consider the dynamic process of human behavior in these applications, we may end up causing unintended damage or negative consequences. Therefore, it is crucial that we use a stochastic multilayer network as a model for understanding human behavior. This approach allows us to gain a deeper understanding of the underlying dynamics and factors that shape human behavior, which in turn can help us make more informed decisions in a variety of settings.

Stochastic multilayer graphs are a type of network model that have several unique features compared to other models such as Single-layer networks, probability networks, and fuzzy networks. The stochastic multilayer graph has some of key characteristics including, Multiple layer representation, time varying interactions, edge weights and uncertainty. Unlike Single-layer networks, which consist of only one layer of nodes and edges, stochastic multilayer graphs have multiple layers.

Each layer represents a different type of relationship or interaction between nodes. Time-varying interactions: Stochastic multilayer graphs allow for time-varying interactions between nodes. This means that the strength and type of relationship between nodes may change over time, allowing for more complex and dynamic modeling of real-world systems. Moreover, stochastic multilayer graphs incorporate uncertainty into the modeling process. This allows for the exploration of different possible outcomes and makes it easier to model systems that are inherently uncertain or unpredictable. In addition, stochastic multilayer graphs can also incorporate edge weights, which represent the strength or intensity of the relationship between two nodes. This allows for a more nuanced modeling of relationships between nodes than simpler network models like probability networks or fuzzy networks.

In this paper, we first review a recent multilayer graph model proposed for social network modeling, and then define some network measures related to stochastic multilayer graphs. After that, these measures are used to design an algorithm based on the Cellular Goore Game (CGG) [11,12] for calculating them under the presumption that the probability distribution functions of the weights connected with the edges are unknown. In CGG-based algorithms, samples from the edges of a stochastic multilayer graph are employed to calculate and estimate the distribution of multilayer measures. To reduce the number of possible samples, learning automata are used to guide the sampling process from the edges of the graph. In CGG-based algorithms, guided sampling is used to take more samples from multilayer graphs that show a high rate of change (e.g., increased user activity) in order to cut down on unnecessary samples from non-promising parts. Experimental evaluation of the proposed algorithms has been conducted on synthetic stochastic multilayer graphs. When sampling from the edges of stochastic multilayer graphs, guided sampling algorithms require significantly fewer samples than standard sampling methods. In the remainder of the paper, the following sections are presented: Section 2 describe definition and notations including brief introduction to some structural network measure for deterministic multilayer graph, an overview of research work related to multilayer networks, learning automata and cellular Goore Game. In Section 3, the proposed measures for stochastic multilayer graphs are described. In Section 4, we describe the proposed algorithm based on Cellular Goore Games for calculating the proposed measures in stochastic multilayer graphs. In Section 5, the simulation results are presented, and in Section 6, the paper is concluded.

2. Definition and notations

A multilayer network $\mathscr{M}$ is represented by $\mathscr{M} = (\diamondsuit, \mathscr{C})$ where $\diamondsuit = \{G_q : q \in \{1,\ldots,M\}\}$ indicates as a graph $G_q = (V_q, E_q)$ called layers of $\mathscr{M}$ that $E_q = V_q \times V_q$ is inter-layer edges and

$$\mathscr{C} = \{E_{qk} \subseteq V_q \times V_k; q,k \in \{1,\ldots,M\}, q \neq k\} \tag{1}$$

$\mathscr{C}$ consist of the edges that connect nodes from different layers G_q and G_k with $q \neq k$. The component of $\mathscr{C}$ is called as crossed layers element, and each element of E_{qk} $(q \neq k)$ is called interlayer edges. Moreover, each layer G_q has a set of nodes that indicates as $V_q = \{v_1^q,\ldots,v_{N_q}^q\}$. Calculating network measures plays an important role in designing network algorithms related to multilayer social network analysis because these measures are used to solve network problems based on the algorithm. In [13] Chinese air transportation network is modeled as a multilayer structure. This model represents commercial airlines through several layers. A weighted network represents the number of seats in a particular airport and also the distance between two airports that are close to one another. In addition to computing the clustering coefficient and average shortest path distance of the airport's networks, it is also possible to select airlines according to their availability and cost-effectiveness. The following are definitions of some measures related to deterministic multilayer networks for background information. We further provide related works by scholars associated with different applications of multilayer social networks. This section ends with a discussion of learning automata and cellular Goore Games (CGG).

2.1. Multilayer deterministic network measures

Identifying and calculating network measures for single-layer networks motivates us to generalize these measures to multilayer networks. However, some definitions make it possible to be defined in multiple ways in multilayer domains. As a result, generalization is subject to a number of complications that do not exist in single-layer networks. To formulate a set of measures for computing quantitative descriptions of multilayer networks, our objective is to develop a set of measures. Some of these measures are applicable to single-layer networks, while others are concerned with multiple layers or their interaction. A traditional measure does not have a specific equivalent for these measures.

2.1.1. Node degree

Many studies have used Degree as a fundamental network measure [14]. In a multilayer network K_{iq} indicates as the degree v_i of layer q and defined by Eq. (2):

$$K_{iq} = \sum_{m=1}^{M} k_i^{[q,m]} \tag{2}$$

where $k_i^{[q,k]}$ indicate the sume of all edges that connect node (i, q) to other nodes in layer k, and computes by Eq. (3):

$$k_i^{[q,k]} = \sum_{j=1}^{N} a_{ij}^{[q,k]} \tag{3}$$

where j is the index of other nodes. In this equation $a_{ij}^{[q,k]}$ indicates 1 if node v_i in layer q adjacent to v_j in layer k and 0 otherwise. Alternatively, the degree v_i in layer q indicate the number of nodes connected to v_i directly. As we can see from the above, $k_i^{[q,q]}$ indicates the degree of the node i in layer q. As the level of centrality is calculated, it is important to recognize that the degree of centrality refers to the degree of human involvement in any "diffusion of information" within the network. Therefore, humans with a high degree of centrality can be involved in many ways.

2.1.2. In-degree

In a directed multilayer network, the In-degree of node i in layer q is denoted by $K_{iq,in}$ and defined as:

$$K_{iq,in} = \sum_{m=1}^{M} k_{i,in}^{[q,m]} \tag{4}$$

where $k_{i,in}^{[q,k]}$ is the sum of incoming edge to node i in layer q to nodes in layer k and computed by the following equation:

$$k_{i,in}^{[q,k]} = \sum_{j=1}^{N} a_{ji}^{[q,k]} \tag{5}$$

where $a_{ji}^{[q,k]}$ is 1 if node v_j in layer q is adjacent to v_i in layer k and 0 otherwise [15]. In a real application a node with a high in-degree is regarded to be prominent or have high prestige among other nodes seek to connect directly.

2.1.3. Out-degree

In a directed multilayer network, the out-degree of node i in layer q is denoted by $K_{iq,in}$ and defined as:

$$K_{iq,out} = \sum_{m=1}^{M} k_{i,out}^{[q,m]} \tag{6}$$

where $k_{i,out}^{[q,k]}$ is the sum of outgoing edge to node i in layer q to nodes in layer k and computed by equation:

$$k_{i,out}^{[q,k]} = \sum_{j=1}^{N} a_{ij}^{[q,k]} \tag{7}$$

where $a_{ij}^{[q,k]}$ is 1 if node v_j in layer q is adjacent to v_i in layer k and 0 otherwise [15]. In a multilayer social network, central nodes in the different layers are of particular interest implies that nodes with many citations have a high In- degree of centrality. In contrast, if a node cites many others, they have high out-degree centrality.

2.1.4. Strength

The strength of node v_i in layer q is defined as follows:

$$\mathscr{S}_{\mathrm{iq}} = \sum_{\mathrm{m=1}}^{\mathrm{M}} \mathrm{s}_{\mathrm{i}}^{[\mathrm{q,m}]} \tag{8}$$

where $s_i^{[q,k]}$ is the sum of adjacent edge weights for weighted network and computed

$$s_i^{[q,k]} = \sum_{\substack{j=1 \\ i \neq j}}^{N} w_{ij}^{[q,k]} \tag{9}$$

where $w_{ij}^{[q,k]}$ is >0 if node v_i in layer q is adjacent to node v_j in layer $k$and its value indicates the weight of edge between node v_i and node v_j [16]. Like degree centrality, a human with high strength centrality is known as a popular with high strength connection to other humans; however, a human with high strength may not necessarily consist of the maximum number of friends. Strength centrality is useful in finding a human affected by the amount of spreading from any information in the network.

2.1.5. Local cross-clustering coeffcient

The local cross-clustering coefficient $\widetilde{C}_i^{[q,k]}$ of a node v_i in layer q with respect to the layer k measure the density of edeges within any two neighbors of node (i,q) belonging to layer k [17]. Therefore, it is a measure of the tendency of a node (i,q) to connect to nodes of layer k that belong to a close community. For a node v_i in layer q with $k_i^{[q,k]} > 1$ the local cross-clustering coeffiecient is given by:

$$\widetilde{C}_i^{[q,k]} = \frac{1}{k_i^{[q,k]}\left(k_i^{[q,k]} - 1\right)} \sum_{r,s,q,k} a_{ir}^{[q,k]} a_{rs}^{[k,k]} a_{si}^{[k,q]} \tag{10}$$

where it is assumed that if $k_i^{[q,k]} = 0$ or $k_i^{[q,k]} = 1$ the local cross-clustering coefficient is zero. Local Cross-Clustering Coefficient provides the information that how densely the nodes in different layers of networks forms the cluster together. In other words, the local clustering coefficient calculates the transitivity characteristics of a multilayer network. And also, most of the friends of a node with a high local cross clustering coefficient can cooperate even if the focal node is removed from the network.

2.2. Modeling of multilayer networks

There have been many deterministic network models presented in different domains for social networks. Researchers have noticed that social networks can be modeled as multilayer graphs. It has been shown that this modeling is applicable to a variety of applications. In transportation applications, networks are often represented as multilayer networks, meaning they have multiple layers [18]. Through the analysis of flight connections at various airports belonging to different airlines, this model aims to investigate the role of specific airports in European air traffic. These connections form the different layers of multilayer data. This multilayer representation facilitates the design of new flight connections and enables airlines to form economic partnerships. The network exhibits several characteristic correlations, such as overlaps between layer boundaries and the degree of individual airports across different layers. Brummitt et al. [19] employed a multilayer network model to investigate the interactions between three power grids in the US network. The grids correspond to the regions of the country that are situated in the Western, Eastern, and Texas regions. It has been demonstrated that adding some connectivity between the layers is beneficial, and provides the largest number of alternative cascade paths. It has been shown that a higher level of interconnectivity might have a negative effect on the ability of neighboring networks to establish large cascades as a result of their increased interconnectivity. It is also significant to note that the addition of new interconnections will increase the capacity and total possible load. This will allow a larger amount of fuel to be transported. Trade networks are investigated as a multilayer network to evaluate the effects of globalization on the economy in [20]. Trade networks have been extensively used to evaluate the current state of economic development in nations and to predict future developments [21,22]. A randomizing algorithm based on hidden variables and randomized network ensembles has also been used to evaluate trade networks [23,24]. The study of a multiplex trade network that includes 162 countries as nodes and 97 layers that have evolved annually from 1992 to 2003, has been conducted in [25,26]. For more clarity, in [26], the authors described the characteristic properties of the different layers. *Barigozzi* et al. [25], suggests that the community structure in each layer can be lost if the aggregated layer does not distinguish between trades of different commodities in order to preserve the community structure.

Ducruet [27] states that the modeling trade network as a multiplex structure is correlated with the diversification of trade ports and their distance from one another. According to [28], the multilayer structure modeling of financial networks may lead to an underestimation of systemic risk related to the estimates based on a single-layer approach. Additionally, a multilayer measure is proposed for evaluating the expected loss in multiplex financial networks with respect to systemic risk computation. In this manner, systemic risk may be compared over time and the extent of expected loss can be determined in accordance with market risk indicators. In [29,30], the Italian Interbank Market is studied from the perspective of a multiplex network. The layers of the multiplex network exhibit a strong correlation between degree and strength of nodes, and degrees are distributed in a highly heterogeneous manner. The results show strong correlations between the centrality measures of large banks across various layers of the Italian Interbank Market [30]. Moreover, a comparison is also made between the results of the interbank network to those of the null-based model, which is constructed by combining independent layers of multiplex modeling in a randomized fashion. There has been a presentation of a multilayer interbank model that can be used to assess system risk [31]. In this network modeling, layers are created by aggregating the securities holdings, short-term and long-term interbank borrowings, and cash of fifty large banks. Based on the input data, a stochastic model is used to generate a multiplex network between the banks that is not known. *Bargigli* et al. [31] described that, how an agent-based contagion process can be implemented based on the proposed model. It has been demonstrated that multiplex interbank networks include relevant nonlinearities in the propagation of stocks that can have significant effects on contagion losses. Using only financial time series, the financial network has been investigated in [32]. As a result of this method, the information is encoded and reduced to a multiplex network between the assets. In this modeling, each layer corresponds to a statistically significant linear, non-linear, tail, and partial correlation between the assets.

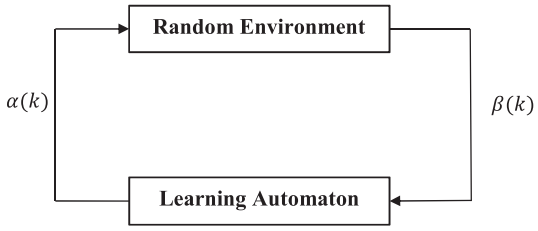


Fig. 1. The communication between the learning automata and random environment.

Using yeast molecular interactions and gene co-expression, a multiplex network analysis was conducted [33]. In the first layer of proteins, genes and their encoded proteins are mapped one to one, forming larger protein complexes. Furthermore, the network can be used to find out which genes are being activated and repressed by particular activators and repressors. A novel algorithm for detecting multi-community environments is presented using a generalized modularity measure called SimMod [33]. The algorithm optimizes this measure to identify community structures. The proposed algorithm outperforms other clustering algorithms by providing improved results even without any prior training on known biological functions at the time of implementation. Cantini et al. [34] developed an algorithm to identify cancer drivers by utilizing multiplex networks derived from gene expression correlation networks that incorporate transcription factor co-targeting, microRNA co-targeting, and protein interaction. The authors have therefore considered four-layer multiplex networks for each type of cancer, where only the first layer (gene co-regulation) is affected by the type of cancer. The four-layer of multiplex network was designed to fine-tune translational and post-translational layers to control gene expression and protein interactions. The multilayer gene communities within the resulting multiplex network are investigated using a consensus clustering approach proposed in [35].

There is a small overlap between the multilayer communities and the communities of the individual layers. By taking into consideration the co-expression layer, a multilayer community provides more information than a co-expression layer-based community. In a different research direction, multilayer network tools are being developed to provide a comprehensive view of a given biological network under a wide variety range of biological conditions. It has been demonstrated that recurrent heavy subgraphs can be formed by simultaneously clustering genes and tissues [36,37]. In [38], conserved modules were identified across protein interaction networks of different species by using a multilayer network perspective. There is also the theoretical challenge of the network alignment problem, which can be formulated as a search for the best fit between two communities of two different networks structure by considering the mapping from one node to another. The spectral clustering algorithm can be used to align communities across the complex networks [38]. In this algorithm, by representing the set of points using hypergraphs, the method is developed based on a generalization of the Perron-Frobenius theorem. It can be shown that a duplex multilayer network is created between a yeast protein interaction network and a human protein interaction network as a result of applying this algorithm. Based on the results of the global assessment, it appears that conservation efforts are being conducted at a high level.

A multi-influence diffusion model MMIC has been proposed for multilayer social networks [39]. In the application of viral marketing, the MMIC model considers multilayer networks as a underlying networks and multiple competing influences propagating among users. According to this model, the author proposes a multilayer fair seed allocation problem. The research presented in [40], epidemic phenomena are proposed to explain the impact of influence maximization on user behavior. On the basis of modeling, the influence of disease spreading is investigated. It has been reported that the probability of a node becoming infected by a diffusion process such as disease spreading is determined by its neighbors or adjacent nodes [40]. An extension of Watts' threshold model is proposed to assess the condition and size of global spreading cascades of innovation [41]. They also assume that if the fraction of infected neighbors in any link type is greater than a certain threshold, a node will change its state to infected. The authors in [42] present a content-dependent threshold model in which each connection type is connected with a relative bias in spreading given content such as a newly launched product. Furthermore, a novel approach has been investigated to determine the importance of a multiplex correlated graph in the maintenance of viral marketing [43]. With tha aid of correlated percolation model, they determined the conditions for viral cascading. A research was carried out to examine the conduct of inactive participants who are commonly referred to as lurkers [44]. Lurkers gain benefit from the community and observe user-generated communications while rarely participating in them. According to the model outlined in [45], the reasons for lurking include environmental factors, personal factors, a lack of commitment to the community, and concerns regarding security. Multilayer networks are able to reveal significant properties of real phenomena that are not detectable through a single-layer network approach. This includes the different roles that industrial sectors play during crisis. In [46] proposed an extended epidemic model and applied it in a multilayer network. They found that infection density and social influence are both associated with self-protection, but their impacts on the epidemic threshold are different. In particular, social influence can only increase the epidemic threshold if its strength is greater than its own threshold. Risk perception cannot affect the epidemic threshold.

In this section, all studies evaluated using multilayer networks as deterministic multilayer graphs. Due to the dynamic nature of many decision-making problems in real-world multilayer networks, some structural and behavioral parameters may change over time. For this reason, deterministic multilayer graphs may not be suitable for modeling such problems. Additionally, most multilayer modeling scenarios assume that the edges of the graph have fixed weights. This is not true when tasks/activities on networks fluctuate over time. Moreover, it is also difficult to capture the continuum of phenomena that occur over time when multilayer networks are modeled using deterministic graphs. A weighted graph analysis of multilayer networks can provide only a snapshot of the real network if the edge weights are assumed to be fixed. As a result of this challenge, stochastic multilayer graphs seem to be a better candidate for describing real-world networks that are unpredictable and time-varying. It should be considered that every characteristic of a stochastic multilayer graph, such as its path, cover, spanning tree, and dominating set, should also be treated as stochastic in nature. In the following, after providing a brief survey of Learning Automata (LA) and Cellular Goore Game (CGG), we define stochastic multilayer graphs in the following subsection and redefine some of their associated network measures.

2.3. Learning automata

The purpose of this section is to provide a brief explanation of LAs and their learning algorithms. Described in the following way is the learning process that takes place when an LA is working. The LA is in this case selecting a randomly selected action from its action set, and then performing the action on the environment to determine its outcome. In response to an action, the environment gives reinforcement signals (rewards or penalties). Following the reinforcement signal provided by the environment, the LA updates its action probability vector and repeats the learning process. It is known as a learning algorithm when an algorithm is used to update the action probability vector. The objective of the learning algorithm is to select an action that maximizes the average reward received from the environment. Fig. 1 illustrates the relationship between learning automata and its random environment.

The LAs can be represented by quadruple $\langle A.B.\mathscr{F}.P(k)\rangle$, where $A =$

$\{\alpha_1 \ldots\ldots \alpha_r\}$, is a finite set of actions, B is the set of all possible inputs or reinforcements to the automation, $\mathscr{I}$ is the learning algorithm for updating action probabilities, and $P(k)$ is the action probability vector at instance k. In order to modify the action probability vector, the learning algorithm uses a recurrence relation.

$$p_i(k+1) = p_i(k) + a(1 - p_i(k)) \tag{11}$$

$$p_j(k+1) = p_j(k) - ap_j(k) \forall j \neq i$$

$$p_i(k+1) = (1-b)p_i(k) \tag{12}$$

$$p_j(k+1) = \frac{b}{r-1} + (1-b)p_j(k) \forall j \neq i$$

Let q_i is the selected action by LAs, at instance k from distribution $q(k)$. In a linear learning algorithm, the updating rule is characterized by Eq. (11) for a favorable response $(\beta = 1)$, and an unfavorable response $(\beta = 0)$. The probability of getting an unfavorable response by action q_i is characterized by $c_i = Pr[\beta = 0|\alpha_i]$. There is no way to determine whether the LA will receive a favorable or unfavorable response. Let a defined as the reward value, which applied for the amount increase of the action probability vector, and b is applied for the penalty value determining the amount the decrease of the action probabilities values. Based on the value of a and b, when the value of a is equal to b, the recursion Eqs. (11) and (12) are called linear reward penalty (L_{R-P}); when a is strictly increasing b the recursion equation is called linear reward-ε penalty ($L_{R-\varepsilon P}$); and finally, when the value of b is equal to zero, the algorithm is called linear reward inaction (L_{R-I}) algorithm. In L_{R-I}. when the environment penalizes an action, the action probability vectors remain unchanged. In many complex and dynamic environment where there exists a great deal of uncertainty or a lack of information, learning has proven to be an important tool for optimizing performance. Due to its simplicity and ability to work in unknown environments, learning automata has contributed significantly to this field, producing numerous works [47,48].

2.4. Cellular Goore-Game (CGG)

A model for optimizing one or more criteria can be developed using the Cellular Goore Games (CGG), introduced in [49]. Generally, CGG can be described as a massive collection of simple objects that interact locally, and this type of model is suitable for modeling. It is described as cellular in the sense that it is composed of cells or points arranged in a structure. A CGG is a network of Goore Games where each node participates in Goore Games with its neighbors, playing by the rules of referees. According to the gains and losses received from the referee adjacent to them, each player selects its optimal action independently. In CGG, players are unaware of how or why other players play or are rewarded or penalized. According to CGG with LA, each player is represented by a learning automaton with two actions which has its action probability vector updated as the learning algorithm progresses. At round k of the game, player i chooses its action $p_i(k)$ from two available actions α_{i1} and α_{i2} according to its action probability vector $p_i(k) = (p_{i1}(k).p_{i2}(k))$. According to the learning algorithm, the chosen action will be either rewarded or penalized based on the average value of the performance criterion determined by neighboring referees (or other criteria). In CGG, the goal is to maximize the sum of $G_i^{'}$s where G_i values, where G_i is the performance criterion for node i.

Definition 1. A Cellular Goore Game with LA can be defined by 5-tuple CGG = (N, P, R, A, G), where

I. $N = (V, E)$ is an undirected network that determines the structure of CGG where V = $\{cell_1.cell_2 \ldots\ldots cell_n\}$ is the set of vertices with $n = |V|$which can be either players or referees or both, and E is the set of edges.
II. P is a subset of V playing the role of players.
III. R is a subset of V playing the role of referees. (the intersection of sets P and R may or may not be empty)
IV. $A = (LA_1, LA_2, \ldots LA_n)$ is a set of learning automata where LA_i is the learning automaton residing in $cell_i$.
V. $G = (G_1, G_2, \ldots G_n)$ is a set of unimodal performance criteria for the referees, where G_i is the performance criterion for $cell_i$.

The aim of CGG is to maximize the total performance criterion, that is, the sum of $G_i^{'}$s where G_i is the performance criterion for *referee i*.

3. Proposed network measure for stochastic multilayer graph

In this section, we first define stochastic multilayer graphs and then define network measures related to stochastic multilayer graph including Node Degree, In-degree, Out-degree, strength and Local Cross-Clustering Coefficient.

3.1.1. Stochastic multilayer graph

A stochastic multilayer network $\mathscr{M}$ can be described by a triple $\mathscr{M} = (\blacklozenge, \mathscr{C}, \mathrm{W})$ where $\blacklozenge = \{G_q : q \in \{1, \ldots, M\}\}$ is a family of graphs $G_q = (V_q, E_q, W_q)$ called layers of $\mathscr{M}$ where $E_q \subseteq V_q \times V_q$ is an interlayer edge and $W_q \in W$ is a matrix in which w_{ij} is random variable connected with edge $e_{ij} \in E_q$ if such edge exists.

$$\mathscr{C} = \{E_{qk} \subseteq V_q \times V_k; q, k \in \{1, \ldots, M\}, q \neq k\} \tag{13}$$

$\mathscr{C}$ is the set of interconnection among nodes of different layers G_q and G_k with $q \neq k$ and $W_{\mathscr{C}} \in W$ is matrix in which $w_{ij} \in W_{\mathscr{C}}$ is random variable connected with edge $e_{ij} \in \mathscr{C}$ if such edge exists.

3.1.2. Stochastic multilayer in-degree

In a stochastic multilayer graph, the In-degree of node i in layer q is denoted by $EK_{iq,in}$ and defined as:

$$EK_{iq,in} = \sum_{m=1}^{M} Ek_{i,in}^{[q,m]} \tag{14}$$

where $Ek_{i,in}^{[q,k]}$ is the sum of the all random variable associated with incoming edge to node i in layer q to nodes in layer k and computed by equation:

$$Ek_{i,in}^{[q,k]} = \sum_{j=1}^{N} \overline{w}_{ji}^{[q,k]} \tag{15}$$

where $\overline{w}_{ji}^{[q,k]}$ is the value of random variable from v_j in layer q to v_i in layer k if there exist.

3.1.3. Stochastic multilayer out-degree

In a directed multilayer network, the out-degree of node i in layer q is denoted by $K_{iq,in}$ and defined as:

$$EK_{iq,out} = \sum_{m=1}^{M} Ek_{i,out}^{[q,m]} \tag{16}$$

where $Ek_{i,out}^{[q,k]}$ is the sum of outgoing edge to node i in layer q to nodes in layer k and computed by equation:

$$Ek_{i,out}^{[q,k]} = \sum_{j=1}^{N} \overline{w}_{ij}^{[q,k]} \tag{17}$$

where $\overline{w}_{ij}^{[q,k]}$ is the value of random variable if node v_j in layer q is

Cellular Goore Game-Based Algorithm for Stochastic Multilayer Network Measure Computation

Inputs: The stochastic multilayer graph $\mathcal{M} = (\mathcal{g}, \mathcal{C}, \mathrm{W})$, Threshold T_{max} , $\mathbf{T_{min}}$

Output: Computed Stochastic Multilayer Network Measure $\hat{\theta}_1^q, \ldots \hat{\theta}_n^q$ where $\hat{\theta}_i^q$ is the network measure each cell i in layer q related to node v_i^q.

Initialization

Create CGG isomorphic to the graph $\mathcal{M}$ by associating with a cell in each node and then assigning a Player/LA in each cell.

Let P is the set of player and P_i^q is the player resided in cell i in layer q.

Let R is the set of referees and R_i^q is the referee resided in cell i in layer q.

Let G is the set of uni-modal performance function and G_i^q is the performance that is assign to cell i in layer q.

Let $\alpha_i(t) = \{\alpha_1^i(t). \ldots . \alpha_n^i(t)\}$ denotes the action set in which $\alpha_j^i(t)$ consist of two actions $\alpha_{j,1}^i$and $\alpha_{j,2}^i$ for LA_j^iin cell v_j^i

Let $p_j^i(t) = (p_{j,1}^i(t), p_{j,2}^i(t))$ be the action probability vector of LA_j^i in cell v_j^i and initialized to $\frac{1}{2}$.

Let $rand(t)$ be the random number at iteration t.

Beginning algorithm

Let t be the iteration number of the algorithm and initially set to 1.

Let $E(p_j^i)$ be the entropy value for the referee in cell v_j^i and initially set to the maximum value

while t< T_{max} **or** Ent > T_{min} **do**

For each $cell_i^q$ in $\mathcal{g}$ **do** in parallel

LA_i^q chooses one of its action $\alpha_i^q(t)$ from two actions $\alpha_{i,1}^q$("YES") and $\alpha_{i,2}^q$ ("NO") according to its action probability vector $p_i^q(t) = (p_{i,1}^q(t), p_{i,2}^q(t))$;

End For

For each $Cell_i^q$ in R, **do** in parallel

$Cell_j^i$ count the fraction of "Yes" corresponding to taking sample from neighboring player nodes

Construct $\hat{\theta}_i^q$ for the each cell i in layer q at round t.

Compute $\varphi_i^q(t)$ based on equation (22)

Compute the mean $\mu_i^q(t)$and standard deviation $\sigma_i^q(t)$ of the sampled node i in layer q and generate reinforcement signal based on equation (23)

End For

For each $Cell_i^q$ in P **do** in parallel

If $\hat{\theta}_i^q$ is In-degree **then**

Let $G_i^q(\hat{\theta}_i^q(t))$ be the the value of referee i in layer q at round t.

If $(rnd(t)) \le G_i^q(\hat{\theta}_i^q(t))$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i1}^q **Then**

LA_i updates its action probability vector based on equation (26);

End IF

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i2}^q**Then**

LA_i updates its action probability vector based on equation (27);

End IF

End If

If $\hat{\boldsymbol{\theta}}_i^q$ is Out-degree **then**

Let $G_i^q(\hat{\theta}_i^q(t))$ be the the value of referee i in layer q at round t.

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i1}^q **Then**

LA_i updates its action probability vector based on equation (26);

End IF

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i2}^q**Then**

LA_i updates its action probability vector based on equation (27);

End IF

End If

If $\hat{\boldsymbol{\theta}}_i^q$ is Strength **then**

Let $G_i^q(\hat{\theta}_i^q(t))$ be the the value of referee i in layer q at round t.

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i1}^q **Then**

LA_i updates its action probability vector based on equation (26);

End IF

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i2}^q**Then**

LA_i updates its action probability vector based on equation (27);

End IF

End If

If $\hat{\boldsymbol{\theta}}_i^q$ is SLCS **then**

Let $G_i^q(\hat{\theta}_i^q(t))$ be the the value of referee i in layer q at round t.

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i1}^q **Then**

LA_i updates its action probability vector based on equation (26);

End IF

If $(rnd(t)) \le G_i^q\left(\hat{\theta}_i^q(t)\right)$ **and** $|G_i^q(\hat{\theta}_i^q(t)) - G_i^q(\hat{\theta}_i^q(t-1))| < \varepsilon_i^q$ and the selected action is α_{i2}^q**Then**

LA_i updates its action probability vector based on equation (27);

End IF

End If

End For

//Calculate LAs Information Entropy

Set Ent = Calculate the average entropy of all LAs in each cells;

Set t = t + 1;

end while

End

Fig. 2. Pseudo code for CGG based algorithm for network measure computations.

adjacent to v_i in layer k.

3.1.4. Stochastic multilayer strength

The stochastic multilayer strength of node v_i of layer q in a stochastic multilayer graph is denoted by Eq. (18):

$$\mathrm{E}\mathscr{S}_{\mathrm{iq}} = \sum_{\mathrm{m}=1}^{\mathrm{M}} \mathrm{Es}_{\mathrm{i}}^{[\mathrm{q,m}]} \tag{18}$$

where $Es_i^{[q,k]}$ is the sum of adjacent random variable associated with node v_i in layer q that is adjacent to node v_j in layer k for $q \in \{1, \ldots, M\}$ and computed by Eq. (19):

$$Es_i^{[q,k]} = \sum_{\substack{j=1 \\ i \neq j}}^{N} \overline{w}_{ij}^{[q,k]} \tag{19}$$

Table 2
The main characteristics of the real networks for experiments.

	# Node	# Edge	# Layer	Description
FLORENTINE [15]	16	35	2	It contains two layers (marriage alliances and business relationships) that describe the relationship between florentine families during the Renaissance, describing the social network as a multilayer social network.
KRACKHARDT HIGH TECH [15]	21	312	3	A multi-layer social networks consist of 3 kinds of relationships between manager of a high-tech company (Advice, Friendship, and "Reports to") that can be described as three different categories of relationships.
AUCS [15]	61	620	5	There are five online and offline layers included in the form of multilayer data, which were obtained at a University research department.
Lazega-Law-Firm [15]	71	2223	3	A study on the social mechanisms of peer cooperation in a corporate law partnership.
London Transportation [15]	369	441	3	An edge is an existing route between a station in London and a node is a train station in London.

where $\overline{w}_{ij}^{[q,k]}$ is a random variable associated with node v_i in layer q v_j in layer k.

3.1.5. Stochastic multilayer local cross-clustering coefficient (SLCC)

The stochastic multilayer local cross-clustering coefficient $\widetilde{EC}_i^{[q,k]}$ of a node v_i in layer q with respect to the layer k is a random variable and computed by Eq. (20):

$$\widetilde{EC}_i^{[q,k]} = \frac{1}{\overline{w}_i^{[q,k]}\left(k_i^{[q,k]}-1\right)} \sum_{r,s,q,k} \overline{w}_{ir}^{[q,k]} \overline{w}_{rs}^{[k,k]} \overline{w}_{si}^{[k,q]} a_{ir}^{[q,k]} a_{rs}^{[k,k]} a_{si}^{[k,q]} \tag{20}$$

where $k_i^{[q,k]}$ is degree of node i in layer q which is connected to nodes in layer k and $a_{ir}^{[q,k]}$ is 1 if node i in layer q is connected to node r in layer k. $\overline{w}_i^{[q,k]}$ is a random variable of node i in layer q which is connected to nodes in layer k.

4. Cellular Goore Game based algorithm for computing network measure

In this section, several algorithms are presented that exploit the Cellular Goore Game to calculate measures in stochastic multilayer graphs. These algorithms are presented when the probability distribution functions of weights connected to edges are unknown. A CGG-based algorithm estimates the distribution of these measures by taking samples from appropriate parts of the edges of the stochastic multilayer graph. To estimate network measures, the proposed CGG-based algorithm uses learning automata to eliminate sampling from edges of the stochastic multilayer graph. The CGG-based algorithm consists of four phases, including 1. Initialization 2. Network measure computation 3. Update the action probability vector 4. Stopping conditions are described below.

4.1. Initialization

Let $\mathscr{M} = (\diamondsuit, \mathscr{C}, \mathrm{W})$ represent a stochastic multilayer graph in which $\diamondsuit = \{G_q : q \in \{1,\ldots,M\}\}$ is a graph $G_q = (V_q, E_q, W_q)$ called a layer of $\mathscr{M}$ and $E_q \subseteq V_q \times V_q$ is an interlayer edge and $W_q \in W$ shows as the weighted matrix in which weight w_{ij}^q applied as a random variable to the edge $e_{ij} \in E_q$ in layer q a possible edge if there is one. Moreover, $\mathscr{C} = \{E_{qk} \subseteq V_q \times V_k; q,k \in \{1,\ldots,M\}, q \neq k\}$ stands for the set of edges among the nodes of different layers G_q and G_k with $q \neq k$ and $W_{\mathscr{C}} \in W$ is matrix in which $w_{i,j}^{qk} \in W_{\mathscr{C}}$ is a random variable associated with edge $e_{i,j}^{qk} \in \mathscr{C}$ a possible edge if there is one. In CGG-based algorithm, each node i which we called to a cell is equipped with a learning automaton with two available actions α_{i1}^q and α_{i2}^q corresponding to "Yes" and "No" where α_{i1}^q ("take sample") and α_{i2}^q ("do not take sample") as described in Goore Game, respectively. Let LA_i^q is a learning automaton residing in $cell_i^q$ with two actions α_{i1}^q ("Yes") and α_{i2}^q ("No"). Moreover, $P_i^q(k) = (p_{i1}^q(k), p_{i2}^q(k))$ is the action probability vector of LA_i^q with an initial value $p_{i1}^q(k) = p_{i2}^q(k) = 0.5$. Let $\widehat{\theta}_i^q(\mathrm{t})$ is the network measure that is calculated by cell i in layer q at iteration t. Moreover, each $cell_i$ in layer q has a uni-modal objective function $G_i^q(\widehat{\theta}_i^q(\mathrm{t}))$. The uni-modal objective function takes the fraction of "*Yes*" action that is selected by LAs to the total number of nodes participating in a sampling process for the network measure under study.

4.2. Network measure computation

During each round of the algorithm, every cell acts as a player and plays the Goore Game simultaneously and independently with its neighboring cells (as players) in order to estimate the distribution of a stochastic multilayer network measure over each round. In each instance, in order to achieve this goal, each learning automaton residing in a cell chooses one of its actions based on its action probability vector. This is because it decides whether or not to take a sample from every edge incident within a cell. Let α_{i1}^q is the selected action that perform by the learning automaton LA_i^q in layer q. Then each referee count the fraction of "Yes" corresponding to α_{i1}^q of its adjacent players to the total number of LAs participating in estimate the distribution of stochastic multilayer network measure. Let $\lambda_i^q(t)$ is the number of adjacent players to $\mathrm{cell_i}$ in layer q selects action q_{i1}. This function counts the action α_{i1}^q by the neighboring players at round t and is defined by Eq. (21):

$$\lambda_i^q(t) = \sum_{i=1}^{K_{iq}} I\{q_i(t) = \alpha_{i1}^q\} \tag{21}$$

where K_{iq} indicates as the degree of node i, and I represent as the indicator function. It should be noted that, since each $\mathrm{cell_i}$ is connected with vertex v_i^q, hereafter(in some cases) vertex v_i in layer q may be referred to as $cell_i^q$ and vice versa. Let $\varphi_i^q(t)$ is the candidate distribution of the stochastic measure at round t that obtained by Eq. (22):

$$\varphi_i^q(t) = \frac{\lambda_i^q(t)}{|\mathrm{d}(v_i^q)|+1} \tag{22}$$

where $\mathrm{d}(v_i^q)$ indicates as the degree of node v_i in layer q. Moreover, $\mu_i^q(t)$ and $\sigma_i^q(t)$ is the mean and standard deviation of the players that select their first action. Then the expected value of $E[\beta_i(t)|\varphi_i^q(t)]$ is denoted by Eq. (23):

$$E[\beta_i^q(t)|\varphi_i^q(t)] = G_i^q\left(\frac{\lambda_i^q(t)}{|\mathrm{d}(v_i^q)|+1}, \mu_i^q(t), \sigma_i^q(t)\right) \tag{23}$$

where

$$G_i^q\left(\frac{\lambda_i^q(t)}{|\mathrm{d}(v_i^q)|+1}, \mu_i^q(t), \sigma_i^q(t)\right) = \frac{1}{\sigma_i^q(t)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\varphi_i^q(t)-\mu_i^q(t)}{\sigma_i^q(t)}\right)^2} \tag{24}$$

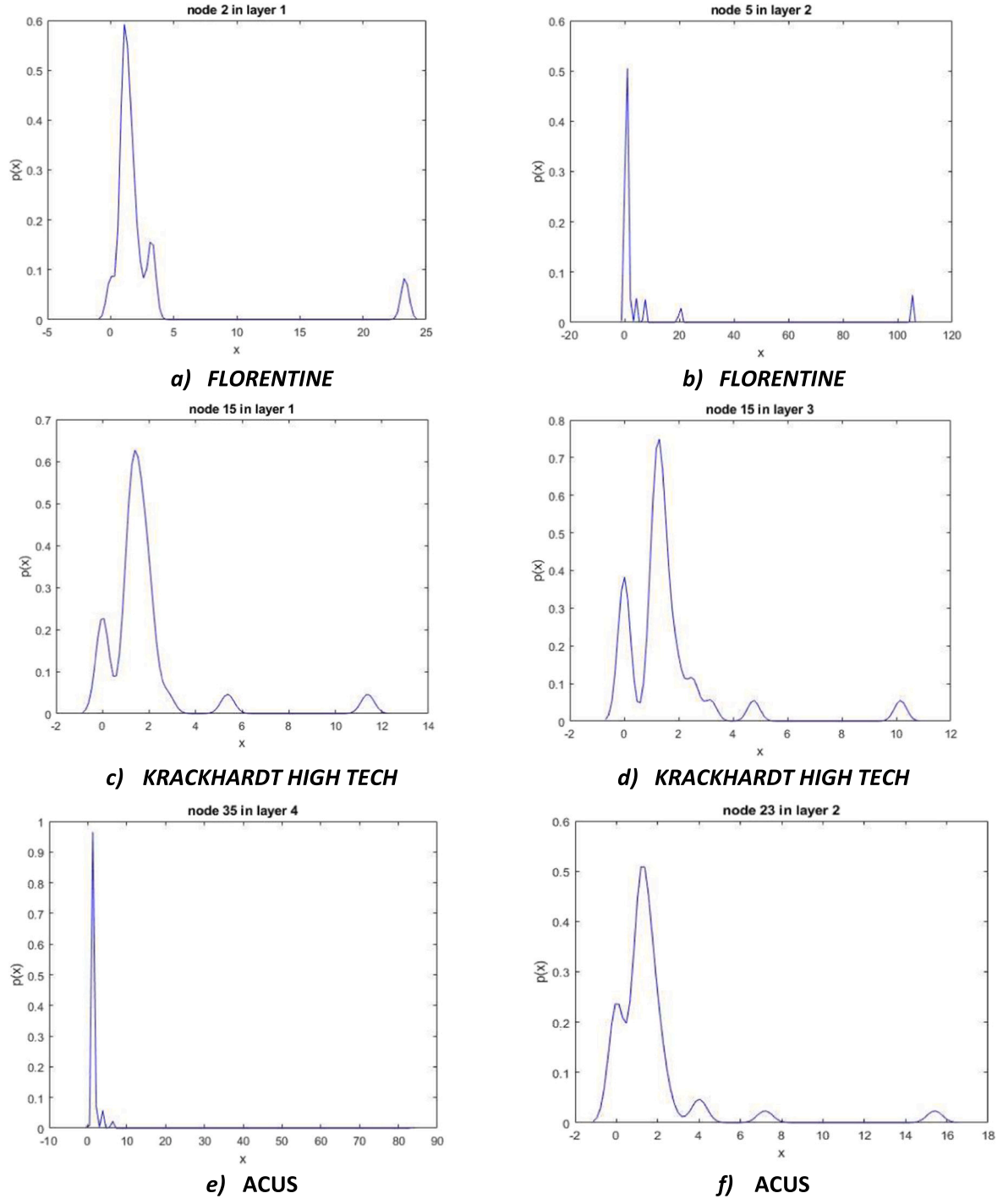


Fig. 3. The distribution of stochastic multilayer degree from some randomly selected nodes.

After $G_i^q\left(\frac{\lambda_i^q(t)}{|d(v_i^q)|+1}, \mu_i^q(t), \sigma_i^q(t)\right)$ is computed by the referee reside in $cell_i$ in layer q. Then, each referee generates reinforcement signal. Let $\beta_i^q(t)$ is defined as the reinforcement signal which is produced by the referee i in layer q, then each players which is adjacent to the referee i update their own selected action.

4.3. Update action probabilities

In this stage, each referee G_i^q computes the difference value of the Kolmograove-Smirnove test (KS-test) based on the Eq. (25):

$$KS(G_i^q(\mathrm{t}), G_i^q(\mathrm{t}-1)) = \max_t |G_i^q(\mathrm{t}) - G_i^q(\mathrm{t}-1)| \tag{25}$$

where $G_i^q(\mathrm{t})$ is the value of referee i in layer q at iteration t. In order to update the selected action of the players, for each player which is adjacent to referee i in layer q, if the value of Eq. (25) is less than ε_i^q, then each adjacent LAs generates a random number independently. If the selected action by a LA_i is equal to α_{i1}^q and the random number is less or equal than the value of the performance function KS then the probability of the action $p_{i1}(t)$ for each adjacent LA_i is updated according to Eq. (26):

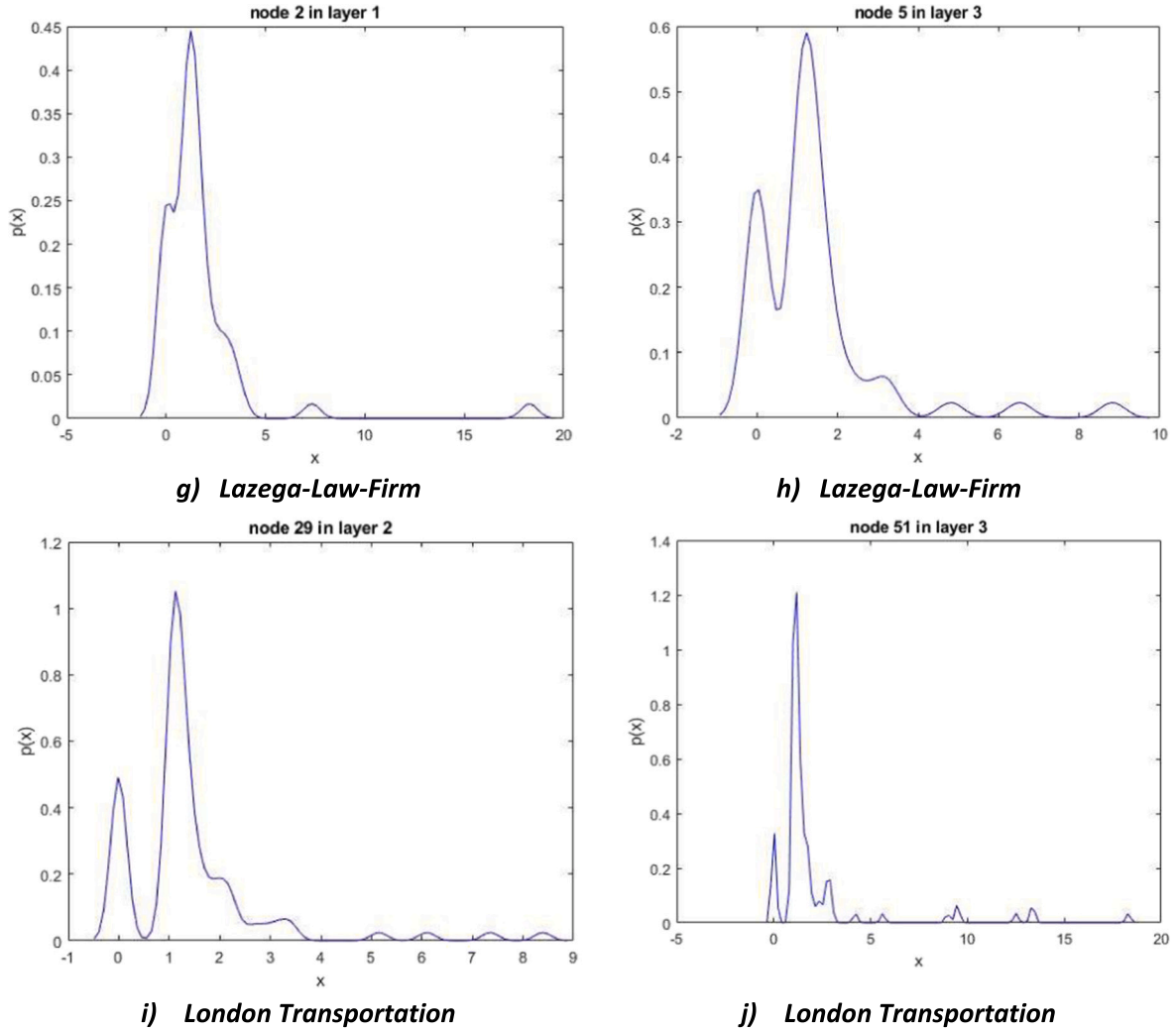


Fig. 3. (*continued*).

$$p_{i1}(t+1) = p_{i1}(t) + \lambda(1 - p_{i1}(t))$$

$$p_{i2}(t+1) = p_{i2}(t) - \lambda p_{i2}(t) \tag{26}$$

where λ is determined the learning rate. On the other hand, if the selected action by each player is equal to α_{i2}^{q} and the random number is less than the value of the performance function *KS* then the probability of the action $p_{i2}(t)$ for each adjacent LA_i is updated based on Eq. (27):

$$p_{i2}(t+1) = p_{i2}(t) + \lambda(1 - p_{i2}(t))$$

$$p_{i1}(t+1) = p_{i1}(t) - \lambda p_{i1}(t) \tag{27}$$

Otherwise, if the random number is greater than the value of either $p_{i1}(t)$ or $p_{i2}(t)$ the action probability vector remains uncahnge.

4.4. Stopping conditions

For all the learning automaton, two criteria may be used for stopping the CGG-based algorithm for network measures computation. The proposed CGG-based algorithm stops, either the number of iterations exceed to T_{max} or entropy reaches a predetermined value. The definition of Entropy is provided as below:

$$E(p) = -\sum_{i \in P} \sum_{j=1}^{2} p_{ij} \, log p_{ij} \tag{28}$$

where p_{ij} is the probability of choosing action j of automaton i. The pseudo-code of the proposed algorithm is given in Fig. 2.

4.5. Strategies

In the proposed algorithm, all learning automata use the same learning rate, which means that the algorithm considers each action with equal importance. The following part of this section discusses various strategies for analyzing the behavior of learning automata in the CGG-based algorithm for network measure computation. These strategies, which are applied to LAs, are described in the next subsection. Moreover, this paper is the first work that is concerned with stochastic multilayer graphs. As a result, we have used the following strategies for comparison.

4.5.1. Strategy 1

The first strategy in the CGG-based algorithm is selecting each action with equal probability. In this strategy, for each algorithm iteration, we proposed that the action be chosen randomly for each learning automaton. Moreover, since each player may participate in more than one referee at the same, a referee may select randomly. In CGG based algorithm, the LAs select their referee for playing GG, and their actions randomly are called Algorithm 1.

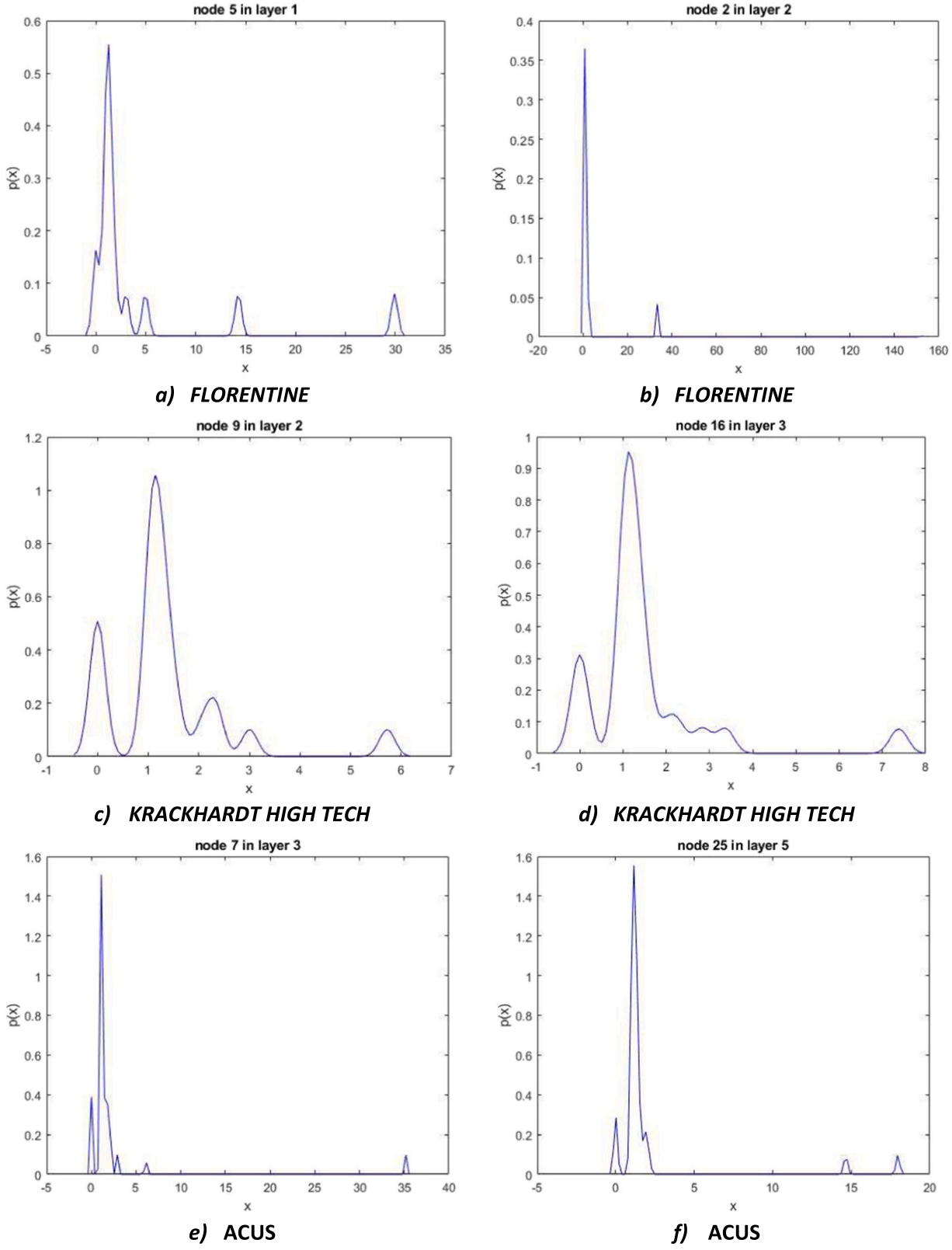


Fig. 4. The distribution of stochastic multilayer in-degree from some randomly selected nodes.

4.5.2. Strategy 2

The second strategy in the CGG-based algorithm is that actions are selected based on the reward value generated by the reinforcement signal. This strategy sets a referee with the maximum degree for each LAs to play GG. In addition, we considered the last recent iterations of the algorithm in which the actions and the action with maximum reward, among others, are selected. This strategy in which a referee is selected based on their maximum degree and the action is selected based

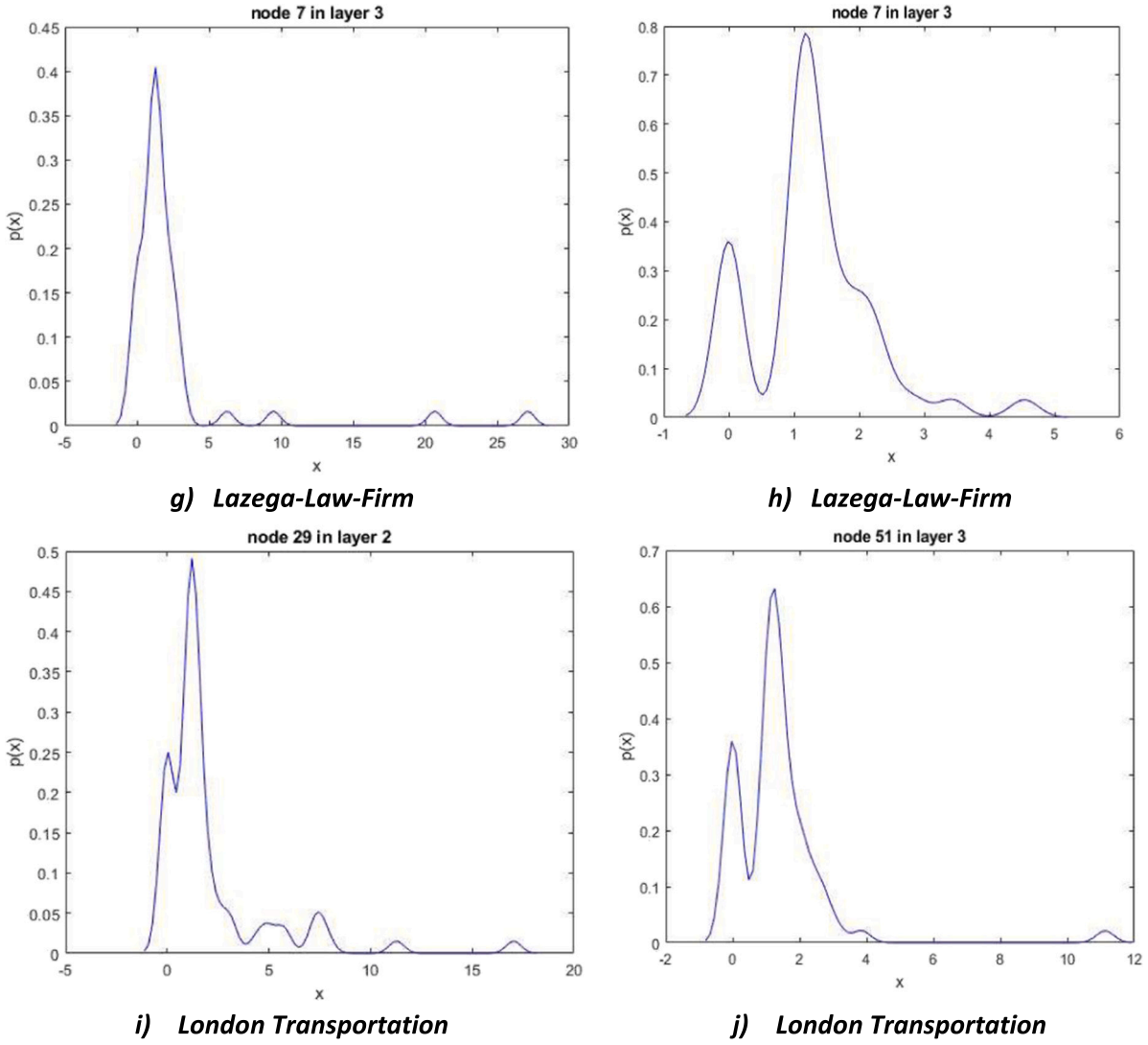


Fig. 4. (*continued*).

on rewards values of k recent iterations is called Algorithm 2.

4.5.3. Strategy 3

The third strategy in the proposed CCG-based algorithm is computed based on average reward for k last iterations for each action. In this strategy for the action selection, the average reward for each action is calculated based on the last k recent iterations, and for the current iteration the action with maximum average reward is selected. Moreover, in this strategy the referee with maximum degree is selected to play GG by LAs. This strategy in which the action is select based on the k maximum average of last recent iterations rewards and the referee are selected based on maximum degree are called Algorithm 3.

4.5.4. Strategy 4

According to the CGG algorithm, the fourth strategy is that each player selects the referee with the highest multilayer clustering coefficient. The referee residing in the cell with the maximum multilayer clustering coefficient is chosen to perform GG. Moreover, the performance of each LA based on their average reward for each action is calculated based on the last k recent iterations that are considered for selecting the action in the next iteration. In Algorithm 4, the action is selected based on the average reward of the k most recent actions and referees with a multilayer cluster co-efficient.

4.5.5. Strategy 5

To improve the speed up of convergence of the learning automata, in [50] proposed the estimator's idea by presenting a Pursuit algorithm indicated called CP_{RP}. In CP_{RP} algorithm, it is assumed that the current action is to be optimal. The current optimal action is determined according to the estimator vector $\widehat{D}$ which is computed as:

$$\widehat{D}_i = \frac{W_i(k)}{Z_i(k)} \tag{29}$$

where $Z_i(t)$ indicates the number of times the action q_i has been selected up to iteration t and $W_i(t)$ indicates the number of times the action q_i has been rewarded up to iteration t. In formula (29) the higher value of $\widehat{D}_i$ means to be current optimal action for iteration t. Based on CP_{RP}, an action is selected from its available action initially. Then, the selected action is either rewarded or penalized; the CP_{RP} only increase the probability of the optimal action. Algorithm 1, in which the LA is pursuit called Algorithm 5.

4.5.6. Strategy 6

The combination of improvement 4 and improvement 5 that LA is pursuit learning automata called Algorithm 6. More clearly, in this combination, each LA selects the referee with high multilayer clustering co-efficient in addition to using pursuit learning algorithm.

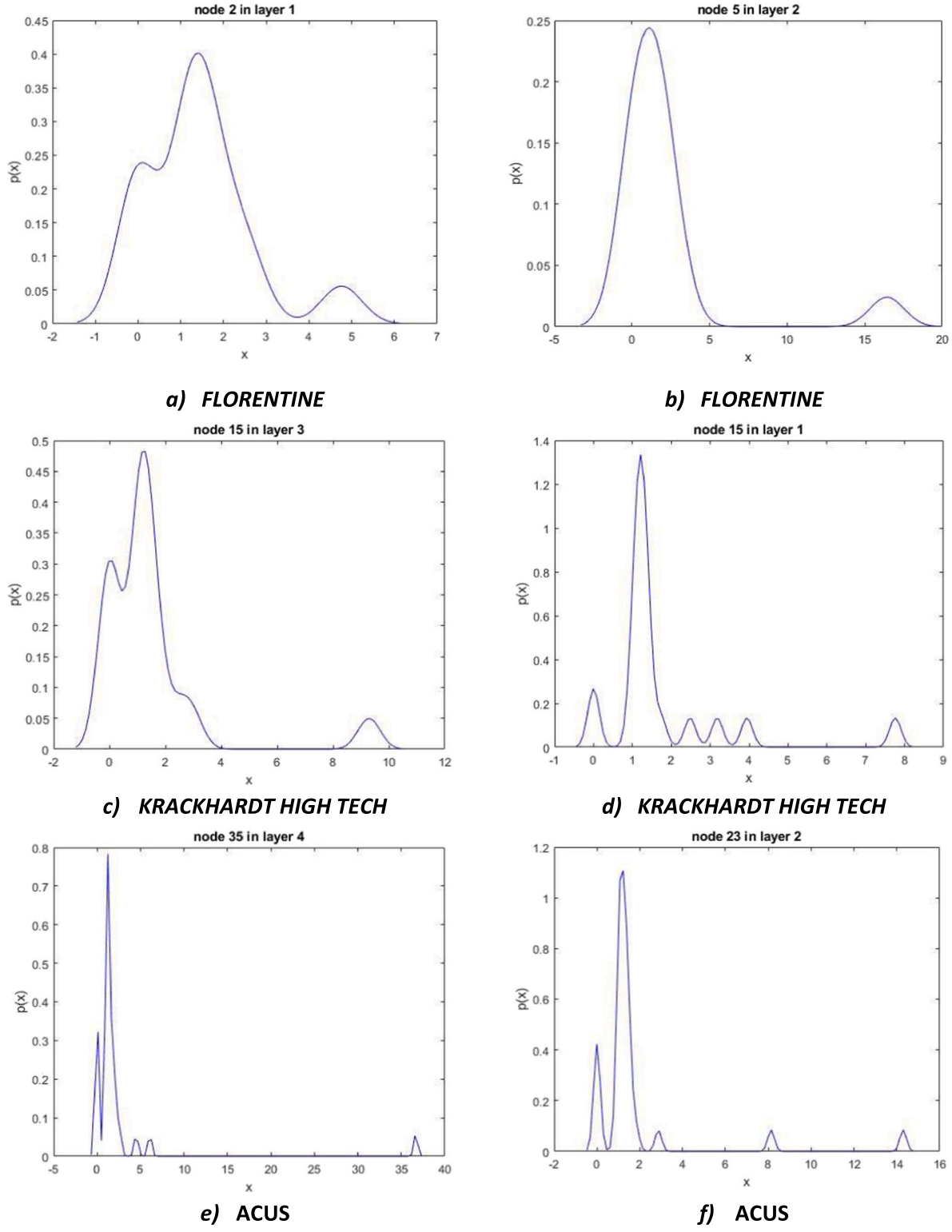


Fig. 5. The distribution of stochastic multilayer out-degree from some randomly selected nodes.

We note that to utilize the proposed algorithm in practical applications, the first step includes constructing a multi-layer stochastic graph using one of the available construction methods, such as crawling from networks. Then, we obtain the input graph from real-world networks, where the behavior of users is represented through different layers of the graph. Users are treated as players, and their relationships are captured through different edges across the layers. Additionally, we associate a random variable that represents the communication rate between users across different layers, which is assigned to the edges of the graph. Then, using an iterative procedure and taking sample with the aid of learning automata, the algorithm attempts to estimate and construct the distribution of criteria for stochastic multi-layer graph in a guided manner.

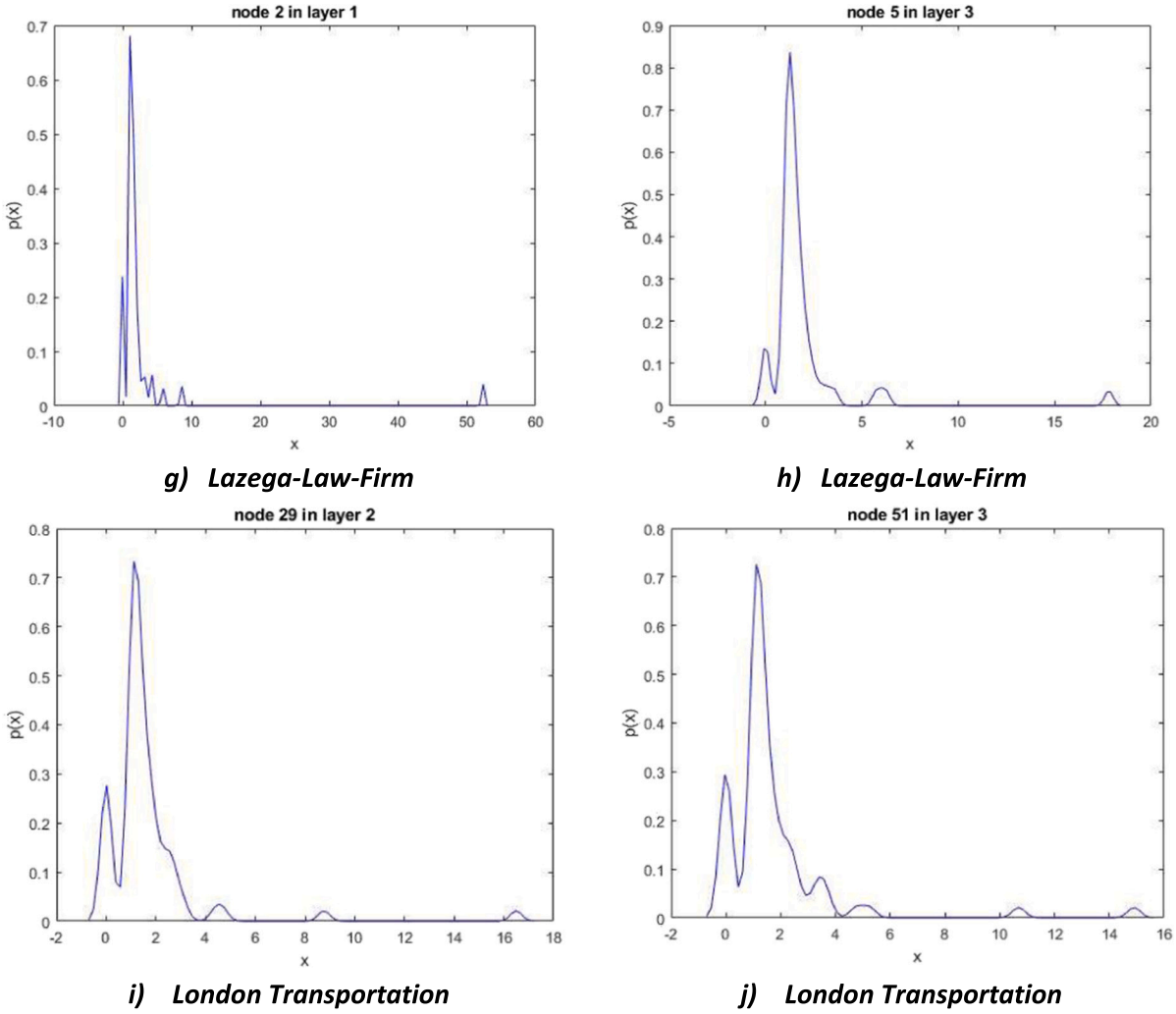


Fig. 5. (*continued*).

The sampling mechanism involves the use of learning automata, which allows the algorithm to sample from the edges of the graph and find suitable samples that accurately represent the distribution of the parameters. Through this process, the algorithm aims to provide accurate estimates of the parameters related to the stochastic multi-layered graph, which is crucial for making informed decisions about the underlying system. The use of learning automata ensures that the distribution estimation mechanism is robust, effective and leading to better outcomes and improved performance.

4.6. Evaluation distance measures

To evaluate the accuracy of estimations by CGG-based algorithms, several distance functions were applied between the real parameters and the estimated parameters. We compare the distribution properties of the proposed algorithms for estimating network parameters using the Kolmogorov-Smirnov distance statistic, skew divergence distance, Pearson's correlation coefficient, and normalized L1 distance. Below is a general description of these distance measures as evaluation criteria.

4.6.1. Kolmogorov–Smirnov statistic

The Kolmogorov-Smirnov test compares a group of samples with a reference probability distribution. KS tests measure the distance between the distribution function of the sample and the cumulative distribution function of the reference distribution. This is done by comparing the empirical distribution function and the cumulative distribution function. The value of this test is between 0 and 1. As closer to zero, both distributions will have a greater similarity. The two distributions will show a greater distinction as closer to the unit. This measure is denoted by Eq. (30)

$$KS(F^*, F) = \max_x |F^*(x) - F(x)| \tag{30}$$

where F^* and F are two CDFs of original and estimated data respectively and x represents the range of the random variable.

4.6.2. Pearson correlation coefficient

One of the appropriate measure to compute the similarity among the estimated parameter value and original parameter value is Pearson correlation Co- efficient and computed as follows.

$$PCC(F(x), F^*(x)) = \frac{n\sum_i f_i f_i^* - \left(\sum_i f_i\right)\left(\sum_i f_i^*\right)}{\sqrt{n\sum_i f_i^2 - \left(\sum_i f_i\right)^2}\sqrt{n\sum_i \left(f_i^*\right)^2 - \left(\sum_i f_i^*\right)^2}} \tag{31}$$

in which $F(x)$ and $F^*(x)$ indicates the values of the real and estimated distribution from stochastic multilayer networks and f_i and f_i^* is the estimated and real parameter of the $F(x)$ and $F^*(x)$ distributions respectively.

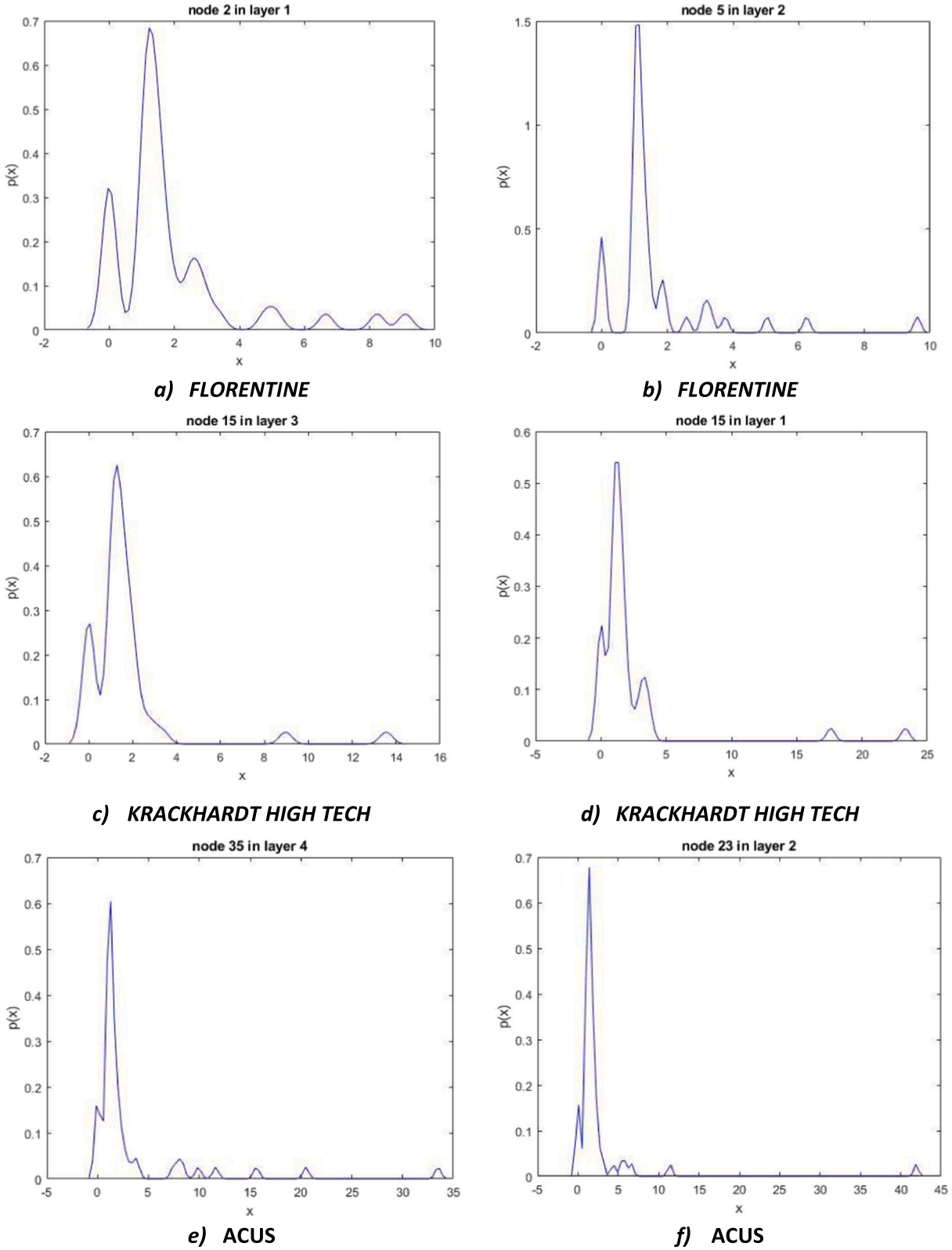


Fig. 6. The distribution of stochastic multilayer strength from some randomly selected nodes.

4.6.3. Normalized Root Mean Square Error (NMSE)

Another measure used for evaluating the sampling algorithm is Normalized Root Mean Square Error (NMSE) for the clustering coefficient which is defined by Eq. (32).

$$NMSE = \frac{\sqrt{(E|F^* - F(x)|)^2}}{F(x)} \tag{32}$$

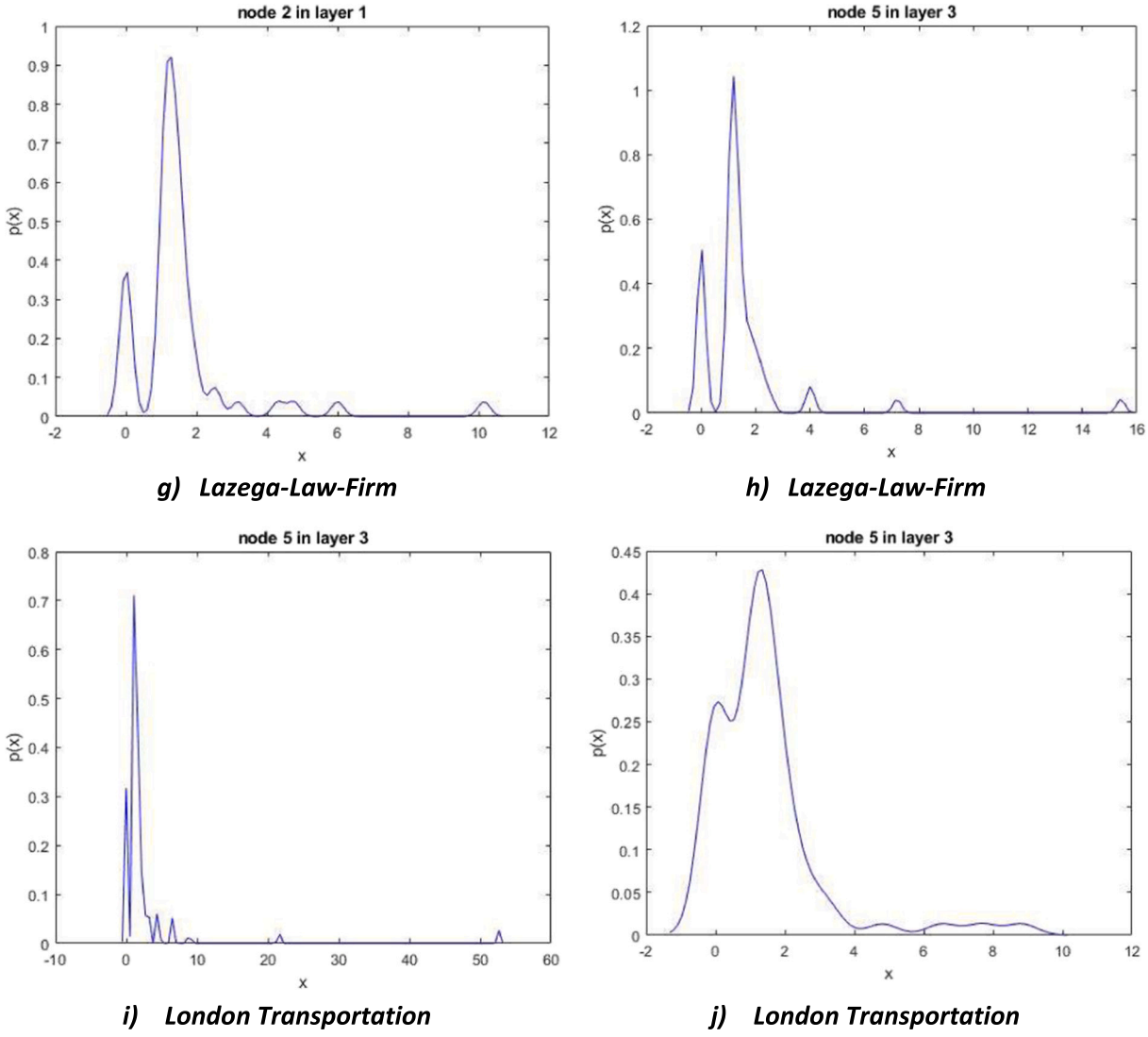


Fig. 6. (*continued*).

where $F(x)$ and $F^*(x)$ indicates the real and estimated distribution from stochastic multilayer networks.

4.6.4. Kullback-Libler divergence

The Kullback–Leibler divergence is a statistical distance that computes how one probability distribution Q is different from a second reference probability distribution *P*. In other words, Kullback-Leibler (KL) divergence measures the similarity between two PDFs $F(x)$ and $F^*(x)$ that do not have continuous support over the full range of values and $q = 0.99$. The Kullback-Leibler (KL) divergence is defined as follows:

$$KL(F(x)\,||F^*(x)) = \sum_x F(x) log \frac{F(x)}{F^*(x)} \tag{33}$$

5. Experiments setup

This section provides the configurations which are used for the experiments. For the experiments of the proposed stochastic network measure algorithm, we have used the real and synthetic multilayer networks in which the edge weights are random variables with Power-law distributions whose value takes randomly from $\alpha = \{1.44, 1.47, 1.53, 1.54, 1.98\}$ and degree exponent k$= \{3, 5, 7, 10, 12\}$. The charachtricts of the real network used for experiment is shown in Table 2. The stopping condition for the proposed algorithms is either the number of iteration t reached $T_{max} = max|v_q| \times 100$ iterations where v_q is the number of nodes in layer q or the value of average information entropy value H(t) reachs lower than T_{min} <0.05 or the difference between the calculated measure in two consecutive iterations based on KS-test becomes lower than $\varepsilon_i^q < 0.01$. For the proposed algorithm, the reinforcement scheme used for updating the action probability vector of learning automata in all experiments are considered L_{R-I} and the learning rate is set to a $= 0.001$.

The proposed algorithm aims to collect information from the network to find reasonable estimates for the network's measurements using fewer samples than that standard sampling methods. The first experiment provides a representation of how one can utilize the approach proposed in the paper to analyze a multilayer social network modeled as a stochastic multilayer graph.

5.1. Experiment I

This experiment is conducted to study the behavior of the CGG-based algorithm for finding stochastic measurements that are proposed for stochastic multilayer graphs. To perform this experiment, we selected two different nodes from different layers randomly and ranked them

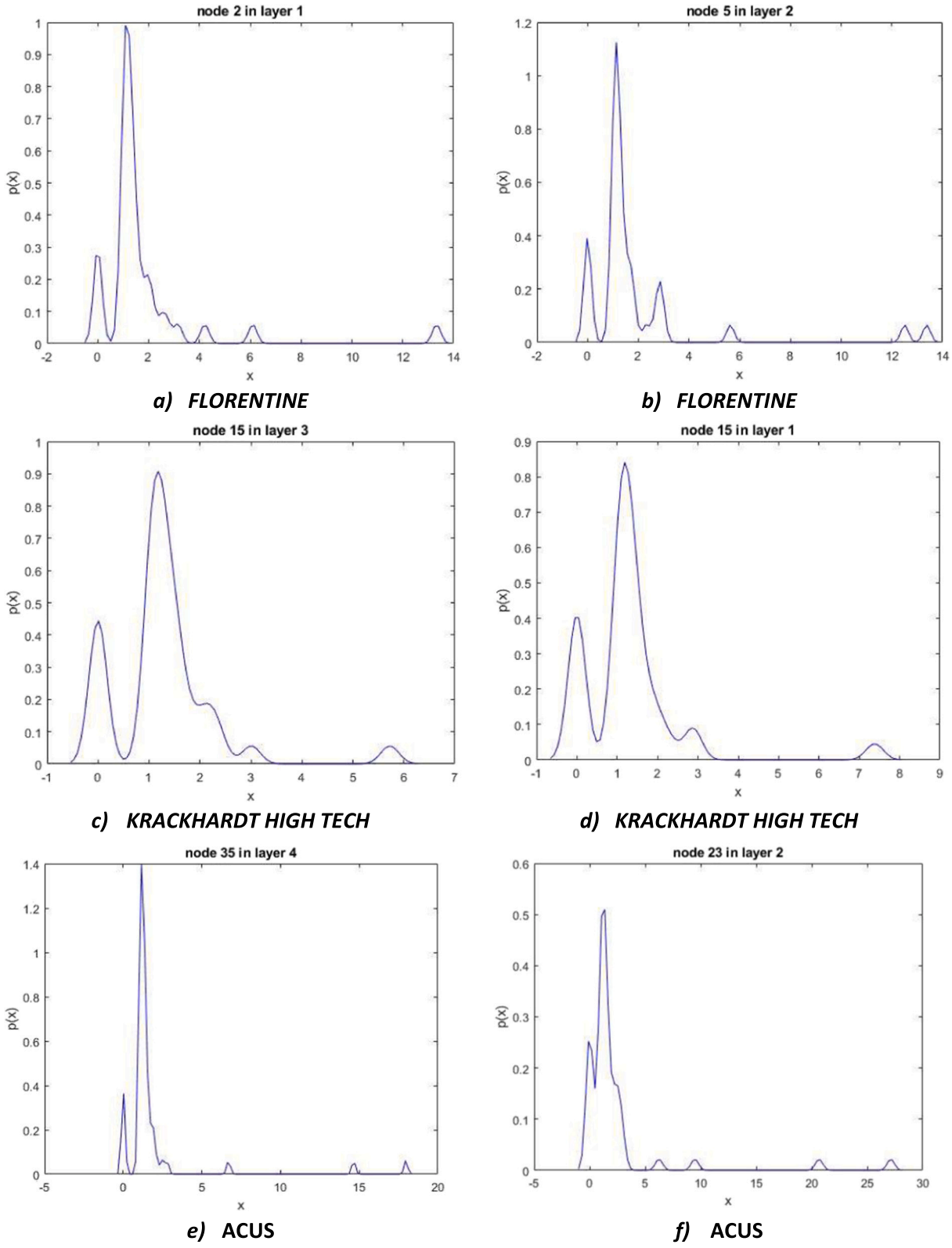


Fig. 7. The distribution of stochastic multilayer local clustering co-efficient from some randomly selected nodes.

based on their high-degree, in-degree, out-degree, strength, and clustering coefficient for each network. The distributions of the stochastic multilayer measures for each of these networks are shown in Figs. 3 to 7. For statistical ranking, we have used Friedman's multi-comparison test with a 95 % significance interval that is reported in Tables 3 to 7. In influence propagation, disease spreading, and the shortest path for rumor spreading are applications of ranking in which the problem of influence maximization is critical. In this case, nodes are ranked based

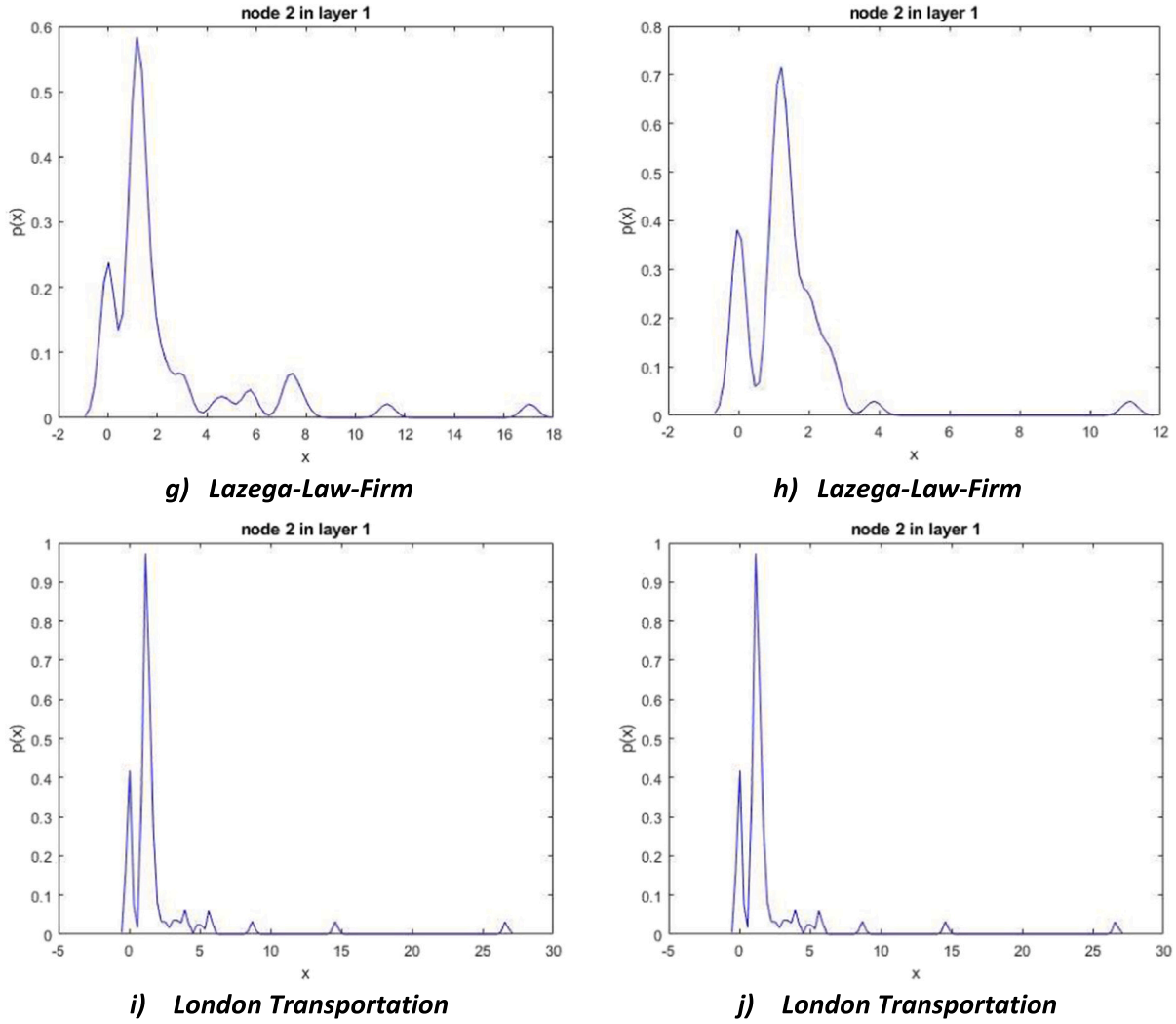


Fig. 7. (*continued*).

Table 3
The average ranking nodes from FLORENTINE stochastic multilayer in terms of different stochastic measures.

	High-degree		In-degree		Out-degree		Strength		Cluetring co-efficient	
	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank
Node 2 in layer 1	1.038	1	1.17	1	1.17	1	1.24	1	3.17	5
Node 3 in layer 2	2.25	2	1.43	2	1.43	2	2.41	2	2.19	2
Node 6 in layer 2	5.58	5	5.01	5	5.01	5	5.71	5	1.05	1
Node 8 in layer 1	2.49	3	3.59	4	3.59	4	2.25	3	2.87	4
Node 11 in layer 1	3.15	4	2.64	3	2.64	3	3.01	4	2.31	3

Table 4
The average ranking nodes from KRACKHARDT HIGH TECH stochastic multilayer graph in terms of different stochastic measures.

	High-degree		In-degree		Out-degree		Strength		Cluetring co-efficient	
	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank
Node 15 in layer 1	2.71	3	2.84	4	2.71	4	2.61	3	2.78	4
Node 15 in layer 3	3.15	4	2.54	2	2.44	2	3.41	4	3.11	5
Node 4 in layer 2	1.24	1	1.34	1	1.24	1	2.01	1	1.52	2
Node 7 in layer 3	1.47	2	2.58	3	2.61	3	2.41	2	1.44	1
Node 8 in layer 3	3.54	5	3.21	5	3.52	5	2.54	5	2.57	3

on their strengths or clustering coefficients, and influence maximization is used to propagate influence. In all Figs. 3 to 7, the x-axis is the values of the random variable and the y-axis is the its probability corresponde to the random variable. From the obtained result, we may conclude that in all Figs. 3 to 7 the network measures under study have a global maximum and several local maximum. The algorithm converges after

Table 5
The average ranking nodes from ACUS stochastic multilayer graph in terms of different stochastic measures.

	High-degree		In-degree		Out-degree		Strength		Cluetring co-efficient	
	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank
Node 35 in layer 4	2.64	4	3.21	5	2.13	2	2.13	3	3.41	5
Node 23 in layer 2	1.05	1	2.17	2	2.18	3	1.52	1	2.12	2
Node 4 in layer 1	1.43	2	1.84	1	1.93	1	1.61	2	2.54	3
Node 18 in layer 3	2.28	3	2.82	4	2.35	4	2.35	4	3.11	4
Node 17 in layer 2	2.71	5	2.62	3	2.64	5	3.11	5	1.24	1

Table 6
The average ranking nodes from Lazega-Law-Firm stochastic multilayer graph in terms of different stochastic measures.

	High-degree		In-degree		Out-degree		Strength		Cluetring co-efficient	
	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank
Node 2 in layer 1	3.41	4	2.52	3	3.05	3	3.43	5	3.41	5
Node 5 in layer 3	2.57	3	2.41	2	2.15	1	2.45	3	3.21	4
Node 17 in layer 2	1.87	1	1.65	1	2.31	2	1.92	2	2.21	2
Node 23 in layer 2	1.96	2	2.54	4	3.24	4	1.75	1	2.25	3
Node 50 in layer 3	3.45	5	3.54	5	3.62	5	3.20	4	2.14	1

Table 7
The average ranking nodes from London Transportation stochastic multilayer graph in terms of different stochastic measures.

	High-degree		In-degree		Out-degree		Strength		Cluetring co-efficient	
	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank	Avg	Rank
Node 2 in layer 1	2.21	2	3.51	5	3.47	5	3.21	3	4.31	5
Node 8 in layer 3	2.12	1	3.41	4	3.39	4	2.11	1	3.41	3
Node 55 in layer 2	2.35	3	3.11	3	3.25	3	2.24	2	3.45	4
Node 23 in layer 3	4.12	5	2.46	2	2.52	2	3.58	5	2.64	2
Node 37 in layer 2	3.11	4	2.41	1	2.48	1	3.54	4	1.23	1

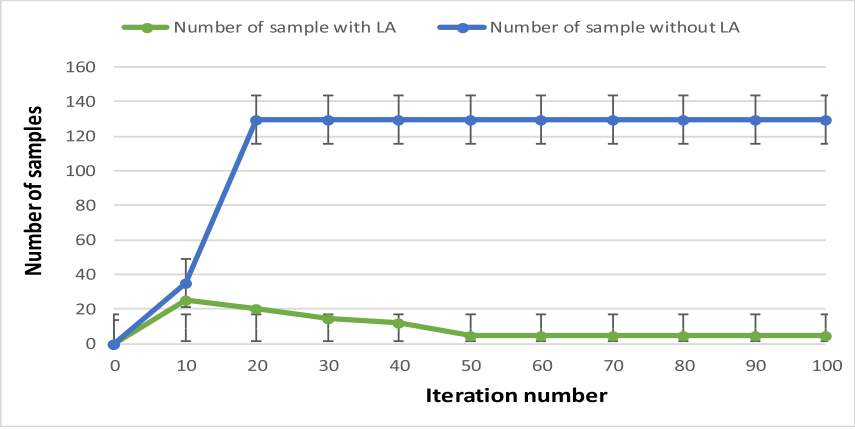


Fig. 8. The plot of average number of sample versus iteration number for stochastic FLORENTINE network.

finding the global maximum.

5.2. Experiment II

This experiment is performed to evaluate the algorithm performance in terms of the number of required samples used by LAs versus the manner in which the LAs are not used. The sampling ratio is computed as the total number of samples taken by LAs over the total number of samples when the LAs are not used. The result of the algorithm is shown in Figs. 8 to 12. To do, some random nodes from each stochastic multilayer network are selected, and then as the algorithm proceeds, the average number of samples which is taken by each of the strategies is reported in Figs. 12 to 14. As can be seen from the result, in all stochastic multilayer graphs, when the algorithm proceeds because of guided sampling, the number of samples taken by LAs gradually reduces and takes fewer samples. While in case the LAs not use, not only is the number of samples not reduced but also the number of samples increases as the algorithm proceeds due to a lack of learning mechanisms.

5.3. Experiment III

This experiment aims to evaluate the number of samples taken by the different strategies applied in the proposed algorithm regarding the number of samples. The result is shown in Tables 8 to 12 for each measure in terms of mean and standard deviation. From the result it shown that in FLORENTINE stochastic multilayer, in terms of high-degree, in-degree, out-degree, strength and clustering co-efficient the algorithms 6 which is used the pursuit learning algorithm and each

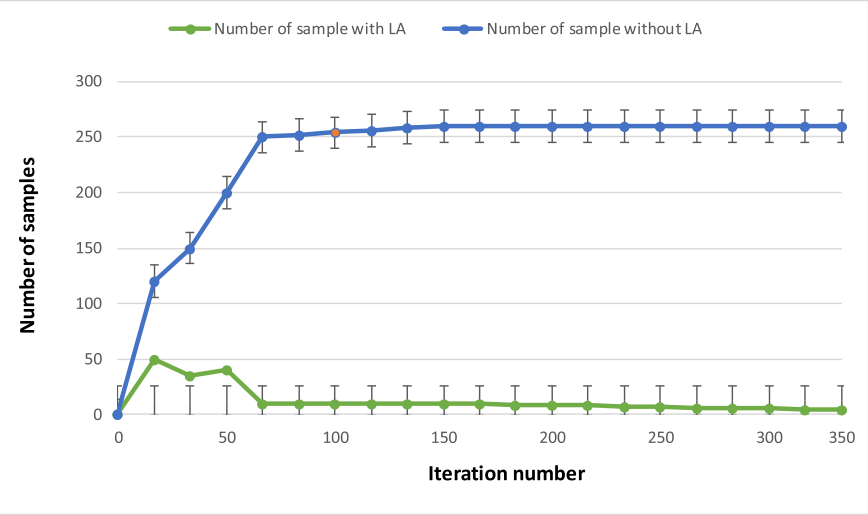


Fig. 9. The plot of average number of sample versus iteration number for stochastic KRACKHARDT HIGH TECH network.

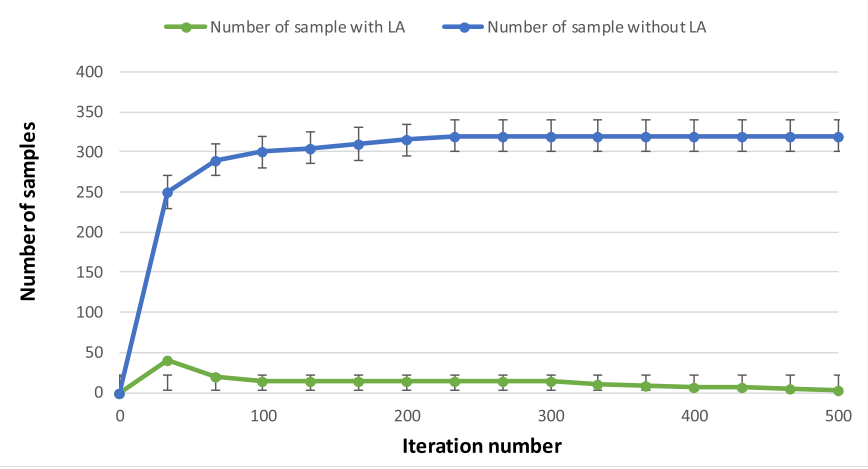


Fig. 10. The plot of average number of sample versus iteration number for stochastic AUCS network.

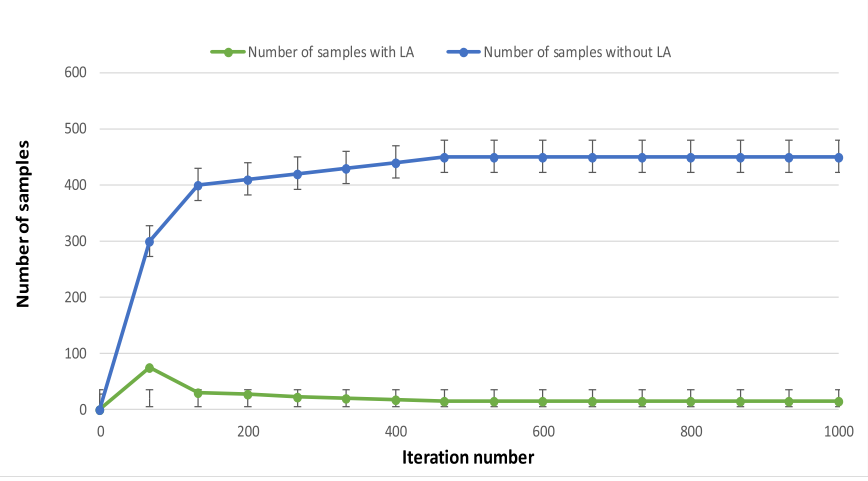


Fig. 11. The plot of average number of sample versus iteration number for stochastic Lazega-Law-Firm network.

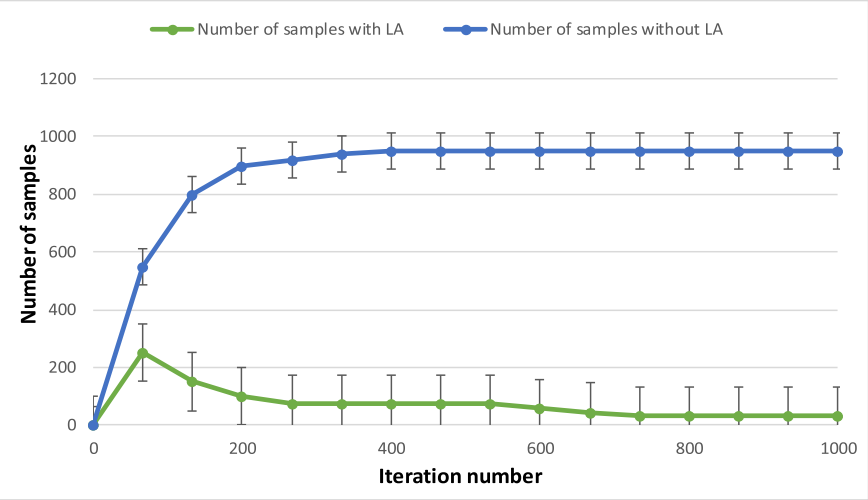


Fig. 12. The plot of average number of sample versus iteration number for stochastic London Transportation network.

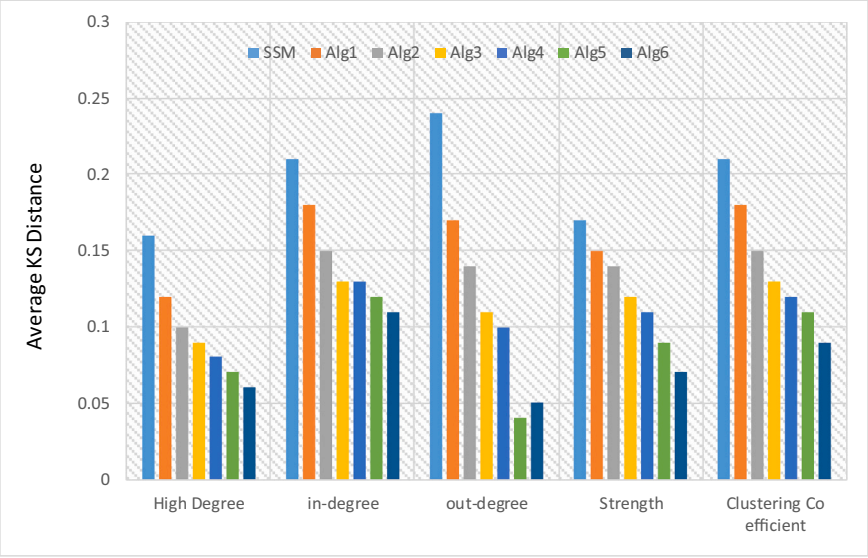


Fig. 13. Comparing results of average KS distance over high-degree, in-degree, out-degree, strength and clustering co-efficient for FLORENTINE network.

players uses the referee with high clustering co-efficient is superior than others. It seems that in all stochastics networks, the Algorithm 6 outperform than other algorithms in terms of number of samples.

5.4. Experiment IV

This test aims to study the behaviors of different algorithms in terms of evaluation criteria. To performs these experiment, the number of required sample by different algorithms are compared with standard sampling methods with 0.05 confidence interval. The results for different measures for each stochastic multilayer networks are reported for average Kolmogorov-Smirnov criteria. Form the results, it is shown that in all learning algorithm the average KS value is outperform than standard sampling methods due to using learning automata for guided sampling. Moreover, among the learning automaton based algorithm, it seems that Algorithm 5 and Algorithm 6 achieve more promising results than other algorithms.

6. Discussion

The role of online users in generating and exchanging information through social networks is significant. In this study, we examine various aspects of multi-layered social networks, including user behavior and the use of different applications within the network. For example, on Twitter, posts, likes, and comments can collectively form a multi-layered network. Concurrent use of different social network platforms by online users, such as Twitter, Facebook, and Instagram, can also contribute to the formation of multilayer social networks with a time-varying nature. Recent research indicates that multilayer social networks can reveal important patterns in user activity that would otherwise be missed by analyzing only one layer of connections. These activities include interacting with friends, responding to their comments, participating in online communities, posting and sharing information content on their friends' pages, visiting their friends' profiles, commenting on a friend's post or liking it. Due to the unpredictable and time-varying nature of social network activities, stochastic multilayer graphs with weighted edges associated with random variables are a better candidate model for analyzing social networks. We obtain a stochastic multilayer graph by capturing snapshot samples of different user profiles at different timestamps, allowing us to capture the time-varying nature of activities from different profiles if we seek to monitor user activity outside the system. However, due to inherent limitations in social networks and a lack of

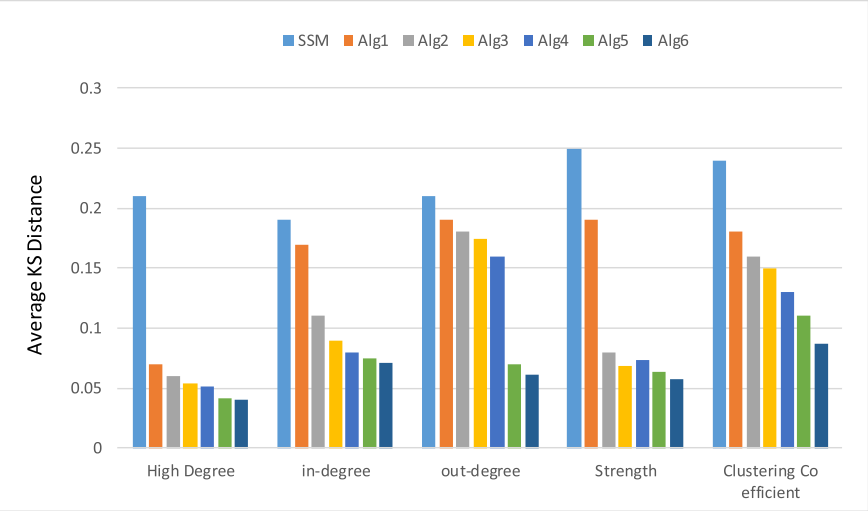


Fig. 14. Comparing results of average KS distance over high-degree, in-degree, out-degree, strength and clustering co-efficient for KRACKHARDT HIGH TECH network.

Table 8
The average ranking nodes from FLORENTINE stochastic multilayer in terms of different stochastic measures.

	High-degree	In-degree	Out-degree	Strength	Cluetring co-efficient
	# of samples	# of samples	# of samples	# of samples	# of samples
Algorithm 1	7.23±0.01	6.41±0.45	7.35±0.15	7.21±1.02	7.55±0.23
Algorithm 2	7.07±0.11	6.55±1.33	7.14±0.41	7.41±0.02	7.42±0.21
Algorithm 3	7.15±0.21	7.33±1.14	6.95±1.32	7.25±0.12	7.32±1.24
Algorithm 4	7.09±0.21	6.19±1.02	6.41±1.42	7.15±0.04	6.87±0.55
Algorithm 5	6.22±1.41	6.11±0.41	6.34±1.31	7.05±1.01	6.42±0.41
Algorithm 6	6.14±1.01	6.09±1.01	6.11±1.08	6.04±0.05	6.33±1.02

access to whole log systems, temporal sampling is necessary to obtain the model. To analyze the stochastic nature of social networks, we take two perspectives. Firstly, as an owner of a social network system who wishes to study user behavior within the system, careful observation reveals that such behavior tends to be uncertain and varies over time. Secondly, when monitoring user activity outside the system, the same limitations require us to employ temporal sampling. As a result, user behavior in social networks exhibits a time-varying parameter and remains uncertain. Since the network under study is considered as a stochastic multilayer graph, we redefined different criteria such as maximum degree, in-degree, out-degree, strength, and local cross-clustering coefficients for the proposed stochastic model. Modeling network as a stochastic multi-layer graph leads to, more information is encoded due to the consideration of details and can provide greater insight into designing effective algorithms that capture the real network characteristic.

When a user enters the system and performs an activity, their actions can be modeled as a multi-layer graph. By displaying these activities in different layers, the graph takes on a time-varying nature that can be captured by a stochastic multi-layer graph model. Suppose the following

Table 9
The average ranking nodes from KRACKHARDT HIGH TECH stochastic multilayer in terms of different stochastic measures.

	High-degree	In-degree	Out-degree	Strength	Cluetring co-efficient
	Samples	Samples	Samples	Samples	Samples
Algorithm 1	12.47±2.51	12.24±2.74	12.57±2.41	13.07±1.24	15.05±1.54
Algorithm 2	12.35±2.24	12.08±1.57	12.15±1.87	12.47±2.74	14.24±1.57
Algorithm 3	12.24±1.95	12.01±1.25	12.12±1.66	12.41±2.52	14.26±1.32
Algorithm 4	12.20±1.80	11.41±1.11	11.34±1.41	11.95±2.05	13.42±2.02
Algorithm 5	11.84±2.05	11.35±1.09	11.29±1.24	11.57±1.75	13.33±1.24
Algorithm 6	11.55±1.56	11.14±0.54	11.21±1.05	11.84±1.54	12.42±1.21

Table 10
The average ranking nodes from AUCS stochastic multilayer in terms of different stochastic measures.

	High-degree	In-degree	Out-degree	Strength	Cluetring co-efficient
	Samples	Samples	Samples	Samples	Samples
Algorithm 1	12.45±2.84	12.47±2.47	12.62±1.85	11.88±3.02	12.48±2.24
Algorithm 2	12.38±2.42	12.35±2.32	12.58±1.75	12.57±2.57	12.27±2.43
Algorithm 3	12.20±2.32	12.24±2.23	12.41±1.24	12.34±2.41	12.13±2.11
Algorithm 4	12.14±2.54	12.12±1.85	12.04±1.11	12.22±2.11	12.02±1.95
Algorithm 5	11.38±2.84	11.95±1.95	12.02±1.57	12.08±1.25	11.75±1.85
Algorithm 6	11.24±2.14	11.45±1.52	12.02±1.41	12.01±0.94	11.55±1.62

Table 11
The average ranking nodes from Lazega-Law-Firm stochastic multilayer in terms of different stochastic measures.

	High-degree	In-degree	Out-degree	Strength	Cluetring co-efficient
	# of samples	# of samples	# of samples	# of samples	# of samples
Algorithm 1	17.85±2.87	19.35±2.01	18.57±2.88	18.57±3.02	22.95±1.89
Algorithm 2	17.55±2.47	18.28±3.24	18.42±2.62	18.24±2.58	21.84±2.32
Algorithm 3	17.25±2.35	17.75±2.84	18.21±2.35	18.41±2.87	20.32±1.72
Algorithm 4	16.87±2.14	17.45±2.54	18.05±1.98	17.25±1.95	20.15±1.52
Algorithm 5	16.57±2.32	17.35±1.95	17.25±2.06	17.12±1.38	19.85±2.25
Algorithm 6	15.34±2.14	17.24±1.54	17.35±1.84	17.05±2.35	18.85±1.85

Table 12
The average ranking nodes from London Transportation network stochastic multilayer in terms of different stochastic measures.

	High-degree	In-degree	Out-degree	Strength	Cluetring co-efficient
	# of samples	# of samples	# of samples	# of samples	# of samples
Algorithm 1	70.62±4.21	73.75±2.12	73.86±2.68	72.95±2.54	75.63±2.38
Algorithm 2	72.24±1.35	72.66±1.45	73.78±1.47	73.52±1.24	74.96±2.58
Algorithm 3	70.55±2.63	71.52±2.75	72.64±2.63	71.68±2.35	73.54±1.95
Algorithm 4	68.95±1.24	71.43±2.56	72.45±2.24	70.75±2.13	73.24±1.72
Algorithm 5	68.12±1.05	71.24±1.36	71.42±1.68	70.32±1.68	72.48±1.35
Algorithm 6	67.02±1.25	70.13±1.25	71.12±1.11	70.24±1.36	72.15±1.36

scenario, where an individual enters a social network, performs an action, and then exits from the network. To analyze the behavior of users in the network, it is imperative to capture their actions. In the case where the network is considered as a single-layer graph, a person who enters the network only once may be recognized as a high-degree node. However, this person may no longer be active in the network in subsequent times. When considering the stochastic nature of user behavior and user activity, this inactive node should not be identified as a high-degree node. Other scenarios are possible as well. For instance, in multi-layer graph modeling, a network user may have a high degree in one layer but not in the others, or it may have a high degree in one snapshot but low in another.

The implementation of stochastic multilayer graphs has the potential to open a new horizon in various domains including social sciences, epidemiology, transportation, and finance as mention few. For example, in the context of social networks, Stochastic multilayer graphs can be used to model the complex relationships between individuals in social networks. For example, researchers can use these graphs to study how people interact with each other at different types of social events, such as parties, bars, and conferences. In epidemiology, stochastic multilayer graphs can be used to model the spread of infectious diseases through populations. Each layer can represent a different mode of disease transmission, such as person-to-person contact, airborne transmission, or surface contamination. Moreover, in transportation networks, stochastic multilayer graphs can be used to model the flow of traffic and goods through transportation networks. Each layer can represent a different mode of transportation, such as road, rail, or air, and the interactions between nodes can represent the movement of vehicles or goods. In addition, in Financial applications, stochastic multilayer graphs can be used to model financial markets and the interactions between different types of assets. Each layer can represent a different type of financial instrument, such as stocks, bonds, or commodities, and the edges can represent the correlations or causal relationships between them.

7. Conclusion

Conventional social network models generally consider connections between users in the form of binary networks. These networks often lack the ability to capture the complexity of relationships between users. As such, there is a need to develop more sophisticated models that can capture the full range of relationships between users. Since people use more than a social network and the behavior of the people in these social networks has an unknown and unpredictable nature. Therefore, in this paper we propose stochastic multilayer graph as a potential model graph to represent the unknown and unpredictable nature of user behavior. This model can incorporate both the complexity of relationships and user behavior uncertainty. Additionally, this model can be used to analyze user behavior's effect on the overall network. Furthermore, we propose six different cellulare goore game based algorithms to estimate some network measures related to stochoastic multilayer graphs. These algorithms model user behavior dynamics in a multilayer network by using learning automata. The proposed algorithm allows for a more accurate representation of the complexities of relationships, as well as user behavior uncertainty. Furthermore, the proposed algorithms analyze the effect of user behavior on network measures, allowing for a better understanding of the network structure. According to our belief, the proposed model can provide an efficient way to analyze online social networks such as human activity and user behavior. It is our intention in the future to generalize our measures and algorithms in relation to various applications of social network analysis, such as multilayer network sampling and finding influential nodes in the context of viral marketing.

Declaration

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that support the findings of this study are openly available in Dickison Magnani Rossi at http://multilayer.it.uu.se/datasets.html [51].

References

[1] Erdös P, Rényi A. On the evolution of random graphs. Publ Math Inst Hung Acad Sci 1960;5(1):17–60.
[2] Kleinberg J. The small-world phenomenon: an algorithmic perspective. In: Proceedings of the thirty-second annual ACM symposium on theory of computing; 2000. p. 163–70.
[3] Barabási A-L, Bonabeau E. Scale-free networks. Sci Am 2003;288(5):60–9.
[4] Leskovec J, Chakrabarti D, Kleinberg J, Faloutsos C, Ghahramani Z. Kronecker graphs: an approach to modeling networks. J Mach Learn Res 2010;11(2).
[5] Clauset A, Newman ME, Moore C. Finding community structure in very large networks. Phys Rev E 2004;70(6):066111.
[6] Virinchi S, Mitra P. Similarity measures for link prediction using power law degree distribution. In: International conference on neural information processing. Springer; 2013. p. 257–64.
[7] Rezvanian A, Meybodi MR. Stochastic graph as a model for social networks. Comput Hum Behav 2016;64:621–40.
[8] Ghavipour M, Meybodi MR. A dynamic algorithm for stochastic trust propagation in online social networks: learning automata approach. Comput Commun 2018; 123:11–23.
[9] Moradabadi B, Meybodi MR. Link prediction in stochastic social networks: learning automata approach. J Comput Sci 2018;24:313–28.
[10] Lagos A-R, Papavassilopoulos GP. Network topology design to influence the effects of manipulative behaviors in a social choice procedure. J Franklin Inst 2022;359 (7):3046–70.
[11] Ameri R, Meybodi MR, Daliri Khomami MM. Cellular Goore Game and its application to quality-of-service control in wireless sensor networks. J Supercomput 2022:1–48.
[12] Khomami MMD, Meybodi MR, Ameri R. Cellular goore game with application to finding maximum clique in social networks. J Comput Des Eng 2022;9(3):966–91.
[13] Hong C, Liang B. Analysis of the weighted chinese air transportation multilayer network. In: 2016 12th World Congress on Intelligent Control and Automation (WCICA). IEEE; 2016. p. 2318–21.
[14] Kivelä M, Arenas A, Barthelemy M, Gleeson JP, Moreno Y, Porter MA. Multilayer networks. J Complex Netw 2014;2(3):203–71.
[15] Kivelä M, Arenas A, Barthelemy M, Gleeson JP, Moreno Y, Porter MA. Multilayer networks. J Complex Netw 2014;2(3):203–71.
[16] Murase Y, Török J, Jo H-H, Kaski K, Kertész J. Multilayer weighted social network model. Phys Rev E 2014;90(5):052810.
[17] Boccaletti S, et al. The structure and dynamics of multilayer networks. Phys Rep 2014;544(1):1–122.
[18] Cardillo A, et al. Emergence of network features from multiplexity. Sci Rep 2013;3 (1):1–6.
[19] Brummitt CD, D'Souza RM, Leicht EA. Suppressing cascades of load in interdependent networks. Proc Natl Acad Sci 2012;109(12):E680–9.
[20] Garcıa-Pérez G, Boguná M, Allard A, Serrano MÁ. Rethinking distance in international trade: world trade atlas 1870–2013. 2015.
[21] Hausmann R, Hidalgo CA, Bustos S, Coscia M, Simoes A. The atlas of economic complexity: mapping paths to prosperity. Mit Press; 2014.
[22] Tacchella A, Cristelli M, Caldarelli G, Gabrielli A, Pietronero L. A new metrics for countries' fitness and products' complexity. Sci Rep 2012;2(1):1–7.
[23] Garlaschelli D, Loffredo MI. Fitness-dependent topological properties of the world trade web. Phys Rev Lett 2004;93(18):188701.
[24] Garlaschelli D, Loffredo MI. Structure and evolution of the world trade network. Phys A: Stat Mech Appl 2005;355(1):138–44.
[25] Barigozzi M, Fagiolo G, Mangioni G. Identifying the community structure of the international-trade multi-network. Phys A: Stat Mech Appl 2011;390(11):2051–66.
[26] Barigozzi M, Fagiolo G, Garlaschelli D. Multinetwork of international trade: a commodity-specific analysis. Phys Rev E 2010;81(4):046104.
[27] Ducruet C. Network diversity and maritime flows. J Transp Geogr 2013;30:77–88.
[28] Poledna S, Molina-Borboa JL, Martínez-Jaramillo S, Van Der Leij M, Thurner S. The multi-layer network nature of systemic risk and its implications for the costs of financial crises. J Financ Stabil 2015;20:70–81.
[29] Bargigli L, Di Iasio G, Infante L, Lillo F, Pierobon F. The multiplex structure of interbank networks. Quant Finance 2015;15(4):673–91.
[30] Bargigli L, di Iasio G, Infante L, Lillo F, Pierobon F. Interbank markets and multiplex networks: centrality measures and statistical null models. In: Interconnected networks. Springer; 2016. p. 179–94.
[31] Montagna M, Kok C. Multi-layered interbank model for assessing systemic risk. 2016.
[32] Musmeci N, Nicosia V, Aste T, Di Matteo T, Latora V. The multiplex dependency structure of financial markets. Complexity 2017;2017.
[33] Bennett L, Kittas A, Muirhead G, Papageorgiou LG, Tsoka S. Detection of composite communities in multiplex biological networks. Sci Rep 2015;5(1):1–12.
[34] Cantini L, Medico E, Fortunato S, Caselle M. Detection of gene communities in multi-networks reveals cancer drivers. Sci Rep 2015;5(1):1–10.
[35] Lancichinetti A, Fortunato S. Consensus clustering in complex networks. Sci Rep 2012;2:336.
[36] Li W, Dai C, Zhou XJ, Tseng G, Ghosh D, Zhou XJ. Integrative analysis of many biological networks to study gene regulation. In: Integrating Omics Data. vol. 68; 2015.
[37] Li W, Liu C-C, Zhang T, Li H, Waterman MS, Zhou XJ. Integrative analysis of many weighted co-expression networks using tensor computation. PLoS Comput Biol 2011;7(6).
[38] Michoel T, Nachtergaele B. Alignment and integration of complex networks by hypergraph-based spectral clustering. Phys Rev E 2012;86(5):056111.
[39] Chen Y, Wang W, Feng J, Lu Y, Gong X. Maximizing multiple influences and fair seed allocation on multilayer social networks. PloS One 2020;15(3):e0229201.
[40] Pastor-Satorras R, Vespignani A. Epidemic spreading in scale-free networks. Phys Rev Lett 2001;86(14):3200.
[41] Brummitt CD, Lee K-M, Goh K-I. Multiplexity-facilitated cascades in networks. Phys Rev E 2012;85(4):045102.
[42] Yağan O, Gligor V. Analysis of complex contagions in random multiplex networks. Phys Rev E 2012;86(3):036103.
[43] Hu Y, Havlin S, Makse HA. Conditions for viral influence spreading through multiplex correlated social networks. Phys Rev X 2014;4(2):021031.
[44] Sun N, Rau PP-L, Ma L. Understanding lurkers in online communities: a literature review. Comput Hum Behav 2014;38:110–7.
[45] Edelmann N. Reviewing the definitions of 'lurkers' and some implications for online research. Cyberpsychol Behav Soc Netw 2013;16(9):645–9.
[46] Huang H, Xu Y, Xing J, Shi T. Social influence or risk perception? A mathematical model of self-protection against asymptomatic infection in multilayer network. Chaos Solit Fractals 2023;166:112925.
[47] Khomami MMD, Rezvanian A, Bagherpour N, Meybodi MR. Minimum positive influence dominating set and its application in influence maximization: a learning automata approach. Appl Intell 2018;48(3):570–93.
[48] Khomami MMD, Rezvanian A, Meybodi MR. A new cellular learning automata-based algorithm for community detection in complex social networks. J Comput Sci 2018;24:413–26.
[49] Khomami, Ameri, Meybodi. Cellular Goore Game and its applications. In: Technical report of the Amirkabir University; 2020.
[50] Narendra KS, Thathachar MA. Learning automata: an introduction. Courier Corporation; 2012.
[51] Dickison ME, Magnani M, Rossi L. Multilayer social networks. Cambridge University Press; 2016.