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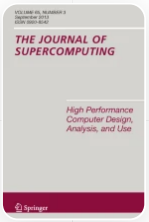
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A cellular goore game-based algorithm for finding the shortest path in stochastic multi-layer graphs

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Abstract

The shortest path problem in stochastic graphs has been extensively studied, with numerous algorithms proposed using various learning automata models. However, the dynamic nature, diverse individual characteristics, and inherent uncertainties of social interactions necessitate the adoption of stochastic multi-layer social network modeling. This approach provides deeper insights into the complex relationships within social networks. When formulated as a stochastic multi-layer graph, key elements such as the shortest path require redefinition to account for these complexities. This paper explores the shortest path problem in stochastic multi-layer graphs and introduces a novel algorithm

based on the Cellular Goore Game (CGG) to address this challenge. The proposed CGG-based algorithm leverages learning automata and extensive edge sampling to determine the optimal path efficiently. By integrating learning automata and selectively sampling from relevant sections of the graph, the algorithm significantly reduces computational complexity. Experimental results on stochastic multi-layer graphs highlight the effectiveness of the proposed algorithm, demonstrating substantial improvements across multiple metrics, including sampling ratio, shortest path ratio, average iterations, and convergence rate.

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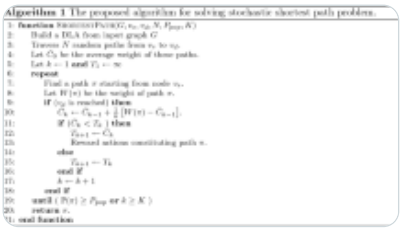
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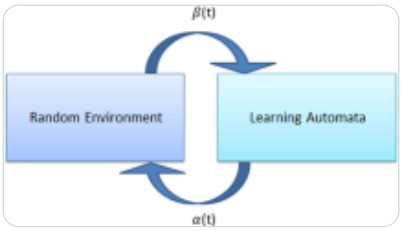
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Data Availability

No datasets were generated or analyzed during the current study.

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Ethics declarations

Conflict of Interest

The authors declare no competing interests.

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Appendix i

This section brings to light two crucial findings that are paramount to this paper for convergence behavior. At first, it demonstrates that Algorithm 1 can converge to the optimal solution by selecting an appropriate learning rate \(a\) when using a learning algorithm \({L}_{R-I}\) at each node. Secondly, it indicates that by selecting an appropriate learning rate, the probability of obtaining the stochastic multi-layer shortest path (SMSP) with the minimum expected weight is higher than \(1-\varepsilon \). We note that the second result supports the connection between the convergence error parameter \(\varepsilon \) and the learning rate applied in the initial proposed algorithm.

Theorem 1: For a given stochastic multi-layer graph, the probability \({q}_{i}\left(k\right)\) of constructing the SMSP \({\omega }_{i}\) at stage \(k\) can be updated using Algorithm 1. It further states that for any small positive value \(\varepsilon \), there

exists a learning rate \({a}^{*}\left(\varepsilon \right)\) between 0 and 1 such that, for any value of \(a\in (0,{a}^{*})\) the following holds:

$$\text{prob}\left[\underset{k\to \infty }{\text{Lim}}{q}_{i}\left(k\right)=1\right]\ge 1-\varepsilon $$

(19)

The method utilized to illustrate Theorem 1 shares a close resemblance to the approach outlined in [38], which is relied on to analyze the LAs' performance in a dynamic environment. The proof comprises several steps, which are outlined concisely. Initially, we prove that the penalty probabilities for all SMSP approach the constant values of the final penalty probabilities as \(k\) becomes sufficiently large (Lemma 1). Moreover, it has been shown that the probability of selecting the SMSP with the lowest weight is a sub-Martingale process for large \(k\). Therefore, it can be deduced that any alterations in the probability of creating the SMSP are consistently non-negative and proved by Lemmas 2 and 3. Finally, with the aid of Martingale convergence theorems, the convergence of the proposed algorithm is proven.

Lemma 1

Let \({\omega }_{i}\) is the SMSP penalized with a probability \({c}_{i}(k)\) at iteration \(k\) and if the \(\underset{k\to \infty }{\text{Lim}}{c}_{i}={c}_{i}^{*}\), then for any value \(\varepsilon \in \left(\text{0,1}\right)\) and any stage \(k\)>\(K(\varepsilon)\), the following statement holds:

$$ - prob\left[{\left| {c_{i}^{*} - c_{i} \left(k \right)} \right| > 0} \right] < \varepsilon $$

(20)

where \(K(\varepsilon)\) indicates the minimum number of stages of algorithm 1 to achieve error rate \(\varepsilon \).

Proof

Before presenting the proof of Lemma 1, we address the convergence of \({c}_{i}\left(k\right)\) to \({c}_{i}^{*}\) (as \(k\) tends to infinity). This discussion follows from the insights derived from Algorithm 1. According to this algorithm, the probability of penalizing SMSP \({\omega }_{i}\) during stage \(k\) is determined by the \({c}_{i}\left(k\right)=\text{prob}[{\overline{W} }_{{\omega }_{i}}>{T}_{k}]\), meaning that each activated LA is penalized based on selected action, such that the average weight of the built SMSP is greater than the optimal threshold \({T}_{k}\). In the initial stages of the algorithm (for small values of k), the optimal threshold \({T}_{k}\) and average weight \({\overline{W} }_{{\omega }_{i}}\) deviate significantly from their true values. This creates a situation where certain SMSPs that are close to optimal are penalized, while SMSPs with higher weights are rewarded. By employing learning automata and sampling from the graph, the algorithm progressively converges \({T}_{k}\) and \({\overline{W} }_{{\omega }_{i}}\) to their precise values. Consequently, the algorithm gains a deeper understanding of the random environment's characteristics with the growth of \(k\). More precisely, as \(k\) increases, the probability of penalizing the optimal or nearly optimal SMSP decreases, while the probability of penalizing other solutions increases. Therefore, when \(k\) reaches sufficiently high values, the probability of penalizing the SMSP \({\omega }_{i}\) approaches its true value \({c}_{i}^{*}\). Assuming \(r\) represents the total count of constructed SMSPs and \({c}_{i}^{*}\) represents the eventual probability value of \({c}_{i}\left(k\right)\) as \(k\) approaches a large value, we can apply the weak law of large numbers may conclude that:

$$\underset{k\to \infty }{\text{Lim}}\text{prob}\left[\left|{c}_{i}^{*}-{c}_{i}\left(k\right)\right|>\varepsilon \right]\to 0.$$

(21)

For every \(\varepsilon \in (\text{0,1})\), there exists a \({a}^{*}(\varepsilon)\in (\text{0,1})\) and \(K\left(\varepsilon \right)<\infty \) such that for all \(a<{a}^{*}\) and \(k>K(\varepsilon)\) we have, \(\text{prob}\left[\left|{c}_{i}^{*}-{c}_{i}\left(k\right)\right|>0\right]<\varepsilon \) and the proof is completed.

Lemma 2

Let \({c}_{j}\left(k\right)\) represent the probability of \([{\overline{W} }_{{\omega } _{j}}\left(k+1\right)>{T}_{k}]\) where \({\overline{W} }_{{\omega }_{j}}\left(k+1\right)\) is the average reward obtained by SMSP \({\omega }_{j}\)(for all \(j=\text{1,2},\dots ,r\)). Additionally, let \({d}_{j}=1-{c}_{j}\) represent the probability of penalizing or rewarding SMSP \({\omega }_{j}\). If \(\underline{q}(k)\) follows from algorithm1, then the conditional expectation of \({q}_{i}(k)\) is defined as

$$ E\left[{q_{i} \left({k + 1} \right)|q\left(k \right)} \right] = \sum\limits_{{j = 1}}^{r} {q_{j} } \left(k \right)\left[{c_{j} \left(k \right)q_{i} \left(k \right) + d_{j} (k)\left({(\prod\limits_{{v_{m} \notin \omega _{i} }} (1 - p_{m}^{1} \left(k \right)\prod\limits_{{v_{m} \in \omega _{i} }} {(p_{m}^{1} \left(k \right))} } \right)} \right] $$

(22)

where

$${p}_{m}^{1}\left(k+1\right)=\left\{\begin{array}{cc}\begin{array}{c}{p}_{m} ^{1}\left(k\right)+a(1-{p}_{m}^{1}\left(k\right))\\ {p}_{m}^{1}(k)(1-a)\end{array}& \begin{array}{c}{v}_{m}^{l}\in {\omega }_{j}\\ {v}_{m}^{l}\notin {\omega } _{j}\end{array}\end{array}\right.$$

(23)

where \({p}_{m}^{1}\) corresponds to the probability that vertex \({v}_{m}^{l}\) as a member of SMSP, at iteration k and \(r\) is the remaining SMSP achieved until iteration \(k\). \({v}_{m}^{l}\in {\omega }_{j}\), if it declares itself as a member of SMSP and \({v} _{m}^{l}\notin {\omega }_{j}\).

Proof:

The proposed algorithm leverages a reinforcement learning scheme, known as \({L}_{R-I}\), to continuously update the probability vector at each stage \(k\). Therefore, when \({\omega }_{j}\) is penalized by the random environment, the construction probability of

the SMSP \({\omega }_{i}\) (i.e., \({q}_{i}(k)\)) remains unchanged. Also, when the candidate SMSP \({\omega }_{j}\) is rewarded, the probability of choosing the vertices that belong to SMSP \({\omega }_{i}\), increasing by a given learning rate with respect to other vertices.

Assuming \({q}_{i}\left(k\right)= \prod_{{v}_{m}^{l}\notin {\omega }_{i}}{p}_{m}^{0}\left(k\right)\prod_{{v}_{m}^{l}\in {\omega }_{i}}{p}_{m}^{1}(k)\) be the probability of constructing SMSP, \({\omega }_{i}\) at iteration \(k\), where \({p}_{m}^{0}\) and \({p}_{m}^{1}\) indicate as the probability with which vertex \({v}_{m}^{l}\) is the member of the shortest path and does not belong to the shortest path, respectively. The conditional expectation of \({q}_{1}(k+1)\), assuming \(\underline{q} (k)\) evolves according to the proposed algorithm, is defined as

$$ E\left[{q_{1} \left({k + 1} \right)|q\left(k \right)} \right] = \sum\limits_{{j = 1}}^{r} {q_{1} } (k)\left[{c_{j} \left(k \right)q_{1} \left(k \right) + d_{j} (k)\left({(\prod\limits_{{v_{m} \notin \omega _{i} }} (p_{m}^{0} \left(k \right)\prod\limits_{{v_{m} \in \omega _{i} }} {(p_{m}^{1} \left(k \right))} } \right)} \right] $$

(24)

where

$${p}_{m}^{0}\left(k+1\right)={p}_{m}^{0}\left(k+1\right)=\left\{\begin{array}{cc}\begin{array}{c}{p}_{m}^{0}\left(k\right)+a(1-{p}_{m}^{0}\left(k\right))\\ {p}_{m}^{0}(k)(1-a)\end{array}& \begin{array}{c}{v}_{m}^{l}\notin {\omega }_{j}\\ {v}_{m}^{l}\in {\omega }_{j}\end{array}\end{array}\right.$$

(25)

$${p}_{m}^{1}\left(k+1\right)=\left\{\begin{array}{cc}\begin{array}{c}{p}_{m}^{1}\left(k\right)+a(1-{p}_{m}^{1}\left(k\right))\\ {p}_{m}^{1}(k)(1-a)\end{array}& \begin{array}{c}{v}_{m}^{l}\notin {\omega }_{j}\\ {v}_{m}^{l}\in {\omega }_{j}\end{array}\end{array}\right.$$

(26)

Since based on the proposed algorithm each LA has two choices, the probability with which vertex \({v}_{m}^{l}\) be a member of SMSP (i.e., \({p}_{m}^{0}\)) is defined as \(1-{p}_{m}^{1}\), we obtain

$$ E\left[{q_{j} \left({k + 1} \right)|q\left(k \right)} \right] = \sum\limits_{{j = 1}}^{r} {q_{j} } (k)\left[{c_{j} \left(k \right)q_{i} \left(k \right) + d_{j} (k)\left({\prod\limits_{{v_{m} \notin \omega _{i} }} {(1 - p_{m}^{1} \left(k \right))} \prod\limits_{{v_{m} \in \omega _{i} }} {(p_{m}^{1} \left(k \right))} } \right)} \right] $$

(27)

where

$${p}_{m}^{1}\left(k+1\right)=\left\{\begin{array}{cc}\begin{array}{c}{p}_{m}^{1}\left(k\right)+a(1-{p}_{m}^{1}\left(k\right))\\ {p}_{m}^{1}(k)(1-a)\end{array}& \begin{array}{c}{v}_{m}^{l}\in {\omega }_{j}\\ {v}_{m}^{l}\notin {\omega }_{j}\end{array}\end{array}\right.$$

(28)

thus, Lemma 2 is proven.

Lemma 3.

Let \({q}_{i}(k)\) denotes the probability of the formation of the SMSP \({\omega }_{i}\), at stage \(k\). The increment in the conditional expectation of \({q}_{i}(k)\) is always non-negative \(\underline{q}(k)\) is updated according to algorithm 1. That is, \(\Delta {q}_{i} >0\).

Let define \(\Delta {q}_{i}\left(k\right)=E\left[{q}_{i}\left(k+1\right)| q\left(k\right)\right]- {q}_{i}\left(k\right)\) from lemma 2. By applying lemma 2, we

observe the following result:

$$\Delta {q}_{i}\left(k\right)=\sum_{j=1}^{r}{q}_{j}\left(k\right)\left[{c}_{j}\left(k\right){q}_{i}\left(k\right)+{d}_{j}(k)(\prod_{{v}_{m}\notin {\omega }_{i}}(1-{p}_{m}^{1}\left(k\right)\prod_{{v}_{m}\in {\omega }_{i}}({p}_{m}^{1}\left(k\right))\right]-{q}_{i}(k)$$

(29)

where

$${p}_{m}^{1}\left(k+1\right)=\left\{\begin{array}{cc}\begin{array}{c}{p}_{m}^{1}\left(k\right)+a(1-{p}_{m}^{1}\left(k\right))\\ {p}_{m}^{1}(k)(1-a)\end{array}& \begin{array}{c}{v}_{m}^{l}\notin {\omega }_{j}\\ {v}_{m}^{l}\in {\omega }_{j}\end{array}\end{array}\right.$$

(30)

Let $\prod_{n\in \left\{\text{0,1}\right\}}{\omega }_{i}{\pi }_{m}^{n}\left(k\right)=(\prod_{{v}_{m}\notin {\omega }_{i}}(1-{p}_{m}^{1}\left(k\right)\prod_{{v}_{m}\in {\omega }_{i}}({p}_{m}^{1}\left(k\right))$, for notational convenience. Therefore, we have

$$\Delta {q}_{i}\left(k\right)=\sum_{j=1}^{r}{q}_{j}\left(k\right)\left[{c}_{j}\left(k\right){q}_{i}\left(k\right)+{d}_{j}(k)\right]-{q}_{i}\left(k\right)$$

(31)

where

$$ \pi _{m}^{n} \left({k + 1} \right) = \left\{ {\begin{array}{*{20}c} {\begin{array}{*{20}c} {p_{m}^{n} \left(k \right) + a\left({1 - p_{m}^{n} \left(k \right)} \right)} \\ {p_{m}^{n} \left(k \right)\left({1 - a} \right)} \\ \end{array} } & {\begin{array}{*{20}c} {v_{m}^{l} \notin \omega _{j} ,~n = 0~or~v_{m}^{l} \in \omega _{j} ,n = 1} \\ {v_{m}^{l}

$$ \notin \omega _{j} ,~n = 1~or~v_{m}^{l} \in \omega _{j} ,n = 0} \\ \end{array} } \\ \end{array} } \right. $$

(32)

where $({\pi }_{m}^{n}(k)$ is the probability with which the vertex $({v}_{m}^{l})$ declare itself as belonging to SMSP and not to belong to the SMSP otherwise. Hence, the by-product of the probability for the vertices that belong to SMSP, we obtain, the probability of constructing, rewarding, and penalizing a SMSP and we have:

$$ \Delta q_{i} \left(k \right) = \mathop \prod \limits_{{\begin{array}{*{20}c} {\omega _{j} } \\ {n \in \left\{ {0,1} \right\}} \\ \end{array} }} E\left[{p_{m}^{n} \left({k + 1} \right)|p_{m} \left(k \right)} \right] - \mathop \prod \limits_{{\begin{array}{*{20}c} {\omega _{j} } \\ {n \in \left\{ {0,1} \right\}} \\ \end{array} }} p_{m}^{n} \left(k \right) $$

(33)

where $({\pi }_{m}^{n}\left(k\right)$ is determined by Eq. (11), $({c}_{m}^{n}\left(k\right)$ indicates as the probability of penalizing the action that is selected by vertex $({v}_{m}^{l})$ at iteration (k) and $({d}_{m}^{n}\left(k\right)=1-{c}_{m}^{n}\left(k\right)$.

$$\Delta {q}_{i}\left(k\right)=\prod_{\begin{array}{c}{\omega }_{j}\\ n\in \{\text{0,1}\}\end{array}}{E[p}_{m}^{n}\left(k+1\right)\left|{p}_{m}\left(k\right)\right]-\prod_{\begin{array}{c}{\omega }_{j}\\ n\in \{\text{0,1}\}\end{array}}{p}_{m}^{n}\left(k\right)$$

(34)

The $(\Delta {q}_{i}\left(k\right)$ be either greater or equal to:

$$\Delta {q}_{i}\left(k\right)\ge \prod_{\begin{array}{c}{\omega }_{j}\\ n\in \left\{\text{0,1}\right\}\end{array}}{(E[p}_{m}^{n}\left(k+1\right)\left|{p}_{m}\left(k\right)\right]-{p}_{m}^{n}\left(k\right))=\prod_{\begin{array}{c}{\omega }

$$_{j}\\ n\in \{\text{0,1}\}\end{array}}{\Delta p}_{m}^{n}\left(k\right)$$

(35)

And

$$\begin{array}{c}{\Delta p}_{m}^{n}\left(k\right)\\ n\in \{\text{0,1}\}\end{array}=a{p}_{m}^{n}(k)\sum_{s\ne n}{p}_{m}^{s}\left(k\right).({C}_{m}^{s}\left(k\right)-{C}_{m}^{n}\left(k\right))$$

(36)

where $({q}_{i}(k)$ is in range of the (0,1) for all $\underline{q}\in {S}_{r}^{0}$), and $({S}_{r}=\{\underline{q}\left(k\right):0\le {q}_{i}(k)\le 1;\sum_{i=1}^{r}{q}_{i}\left(k\right)=1\}$). Also, $({S}_{r}^{0}$) indicates the interior of $({S}_{r}$). We may conclude that, $({p}_{m}^{n}(k)\in (\text{0,1})$) for all, $(n$). Since action $({\alpha }_{m}^{n}$) is the action with minimum probability which can be selected by automation $({A}_{m}$), it is shown that $({C}_{m}^{{S}^{*}}- {C}_{m}^{{n}^{*}}>0$), for all $(s\ne n,$) where $({C}_{m}^{{S}^{*}}$) is the final value to which the penalty probability $({C}_{m}^{{S}^{*}}$) is converged. Lemma 1 shows that when $(k$) is a large value, the difference between $({C}_{m}^{{S}^{*}}$) and $({C}_{m}^{n}$) is positive. Hence, for a large value of k, the right side of the equation contains only non-negative quantities. So, we can confidently say that.

$$\prod_{\begin{array}{c}{\omega }_{i}\\ n\in \left\{\text{0,1}\right\}\end{array}}a.{p}_{m}^{n}(k)\sum_{s\ne n}{p}_{m}^{s}\left(k\right).({c}_{m}^{s}\left(k\right)-{c}_{m}^{n}\left(k\right))\ge 0$$

(37)

And from Eq. (12), we have

$$\Delta {q}_{i}(k)\ge \prod_{\begin{array}{c}{\omega }_{i}\\ n\in \left\{\text{0,1}\right\}\end{array}}a.{p}_{m}^{n}(k)\sum_{s\ne n}{p}_{m}

^{s}\left(k\right).({c}_{m}^{s}\left(k\right)-{c}_{m}^{n}\left(k\right))\ge 0$$

(38)

That concludes the proof of this lemma ■.

Corollary 1.

The set of unit vectors in \({S}_{r}-{S}_{r}^{0}\), where \({S}_{r}^{0} =\{\underline{q}\left(k\right):{q}_{i}\left(k\right)\in \left(\text{0,1}\right);{\sum }_{i=1}^{r}{q}_{i}\left(k\right)=1\}\), represents the set of entire absorbing barriers of the Markov process \({\{\underline{q}\left(k\right)\}}_{k\ge 1}\).

Proof:

This corollary has already been demonstrated in [25] ■.

Let \({\Gamma }_{i}(q)\) represent the probability of the algorithm converging to the unit vector \({e}_{i}\) using the initial probability vector \(\underline{q}\). It can be defined as \({\Gamma }_{i}\left(q\right)=prob\left[{q}_{i}\left(\infty \right)=1|\underline{q}\left(0\right)=\underline{q}\right]=prob[{q}^{*}={e}_{i}|\underline{q}\left(0\right)=\underline{q}]\). Moreover, let \(C\left({S}_{r}\right):{S}_{r}\to \mathcal{R}\) be the state space of all real-valued continuously differentiable functions with bounded derivative defined on \({S}_{r}\), where \(\mathcal{R}\) is the real line. If \(\psi \left(.\right)\in C\left({S}_{r}\right),\) the operator \(U\) is defined by.

$$ U\psi \left(q \right) = E[\psi q\left({k + 1} \right)|q\left(1 \right) = q], $$

(39)

where \(E[.]\) represents the mathematical expectation. It has been shown in [25] that operator \(U\) is linear, and as the expectation of a non-negative function remains non-negative, operator \(U\) preserves the non-negative functions. In other words, \(U\psi (q)\ge 0\) for all \(q\in {S}_{r}\), if \(\psi (q)\ge 0\). If operator \(U\) is repeatedly applied n times (for all n > 1), we have

$${U}^{n-1}\psi \left(q\right)=E[\psi q(k+1)|q\left(1\right)=q]$$

(40)

A function \(\psi \left(q\right)\) is referred to as super-regular (sub-regular) if and only if \(\psi \left(q\right)\ge U\psi \left(q\right)(\psi \left(q\right)\le U\psi \left(q\right))\) for all \(q\in {S}_{r}\). It has also been demonstrated in [65] that \({\Gamma }_{i}(q)\) is the sole continuous solution to the equation \(U{\Gamma }_{i}\left(q\right)={\Gamma }_{i}(q)\), along with specific boundary conditions.

$${\Gamma }_{i}\left({e}_{i}\right)=1$$

$${\Gamma }_{i}\left({e}_{j}\right)=0, j\ne i$$

(41)

Let's define \({\phi }_{i}\left[x,q\right]\) as \(\frac{{e}^{-x{q}_{i}/a}-1}{{e}^{-x/a}-1}\), where \(x>0\) is chosen. The function \({\phi }_{i}\left[x,q\right]\) belongs to \(C({S}_{r})\) and fulfills the mentioned boundary conditions.

Theorem 2:

Let \({\psi }_{i}(.)\in C({S}_{r})\) is sub-regular with the same boundary conditions, then

$${\psi }_{i}(q)\ge {\Gamma }_{i}(q)$$

(41)

For all \(q\in {S}_{r}\).

Proof.

Let \({\psi }_{i}(.)\) be sub-regular, that is,

$${\psi }_{i}\left(q\right)-{U\psi }_{i}\left(q\right)\ge 0$$

(43)

For all \(q\in {S}_{r}\). Applying \(U\) again to the l.h.s. of Eq. (42) and recognizing the fact that \(U\) is linear and preserves non-negative functions,

$${U\psi }_{i}\left(q\right)- {{U}^{2}\psi }_{i}\left(q\right)\ge 0$$

(44)

A repeated application of the operation yields

$${U\psi }_{i}\left(q\right)\ge {{U}^{2}\psi }_{i}\left(q\right). \ge \dots \ge {{U}^{\infty }\psi }_{i}\left(q\right).$$

(45)

We know that \(q\left(n\right)\) converges w.p.1. It follows that \({{U}^{\infty }\psi }_{i}\left(q\right)\) exists and is unique. Moreover,

$${{U}^{\infty }\psi }_{i}\left(q\right)=E\left[{\psi }_{i}\left(q\left(\infty \right)\right)|q\left(0\right)=q\right]=\sum_{j=1}^{r}{\psi }_{i}\left({e}_{j}\right)\text{Pr}\left[q\left(\infty \right)={e}_{j}|q\left(0\right)=q\right]=\text{Pr}[q\left(\infty \right)={e}_{i}|q\left(0\right)=q]$$

(46)

Since \({\psi }_{i}\left({e}_{j}\right)=0\) for \(j\ne i\) and \({\psi }_{i}\left({e}_{j}\right)=1\).

Thus, by inspection of the r.h.s of Eq. (44), we have

$${{U}^{\infty }\psi }_{i}\left(q\right)={\Gamma }_{i}(q)$$

(47)

From (42), (44) and (46) it follows that \({\psi }_{i}(q)\ge {\Gamma }_{i}(q)\). ■

In what follows, we show that \({\phi }_{i}\left[x, q\right]\) is a sub-regular function. Therefore, \({\phi }_{i}\left[x,q\right]\) qualifies as a lower bound on \({\Gamma }_{i}\left(q\right)\). Since super and sub-regular functions are closed under addition and multiplications by a positive constant, and if \(\phi (.)\) is super regular then \(-\phi (.)\) is sub-regular, it follows that \({\phi }_{i}[x,q]\) is sub-regular if and only if

$${\theta }_{i}\left[x,q\right]={e}^{\frac{-x{q}_{i}}{a}}$$

(48)

is super-regular. We now determine the conditions under which \({\theta }_{i}\left[x,q\right]\) is super-regular. From the definition of operator \(U\) given in Eqs. 38 and 30, we have

$$ \begin{aligned} & U\theta _{i} \left[{x,q} \right] = E\left[{e^{{\frac{{ - xq_{i} \left({k + 1} \right)}}{a}}} |q\left(k \right) = q} \right] \\ & \quad \quad \quad \quad \quad = \sum\limits_{{j = 1}}^{r} {q_{j} } d_{j}^{*} e^{{ - \frac{x}{a}[\prod _{{\begin{array}{*{20}c} {v_{m} \notin \omega _{j} ,n = 0} \\ {or} \\ {v_{m} \in \omega _{j} ,n = 1} \\ \end{array} }} (p_{m}^{n} + a(1 - p_{m}^{n}))]}} \\ & \quad \quad \quad \quad \quad + \sum\limits_{{j = 1}}^{r} {q_{j} } d_{j}^{*} e^{{ - \frac{x}{a}[\prod _{{\begin{array}{*{20}c} {v_{m} \notin \omega _{j} ,n = 1} \\ {or} \\ {v_{m} \in \omega _{j} ,n = 0} \\ \end{array} }} (p_{m}^{n} (1 - a))]}} \\ & = [q_{i} d_{i}^{*} e^{{ - \frac{x}{a}\left({q_{i} + a\left({1 - q_{i} } \right)} \right)}} \\ & \quad \quad \quad \quad \quad + \sum\limits_{{j \ne i}} {q_{i} } d_{i}^{*} e^{{ - \frac{x}{a}[\prod _{{\begin{array}{*{20}c} {v_{m} \notin \omega _{j} ,n = 0} \\ {or} \\ {v_{m} \in \omega _{j} ,n = 1} \\ \end{array} }} (p_{m}^{n} + a(1 - p_{m}^{n}))]}} + \sum\limits_{{j = 1}}^{r} {q_{j} } d_{j}^{*} e^{{ - \frac{x}{a}[\prod _{{\begin{array}{*{20}c} {v_{m} \notin \omega _{j} ,n = 1} \\ {or} \\ {v_{m} \in \omega _{j} ,n = 0} \\ \end{array} }} (p_{m}^{n} (1 - a))]}} \\ \end{aligned} $$

(49)

In the context of large values of \(k\), we use \({d}_{j}^{*}\) to represent the final value that the reward probability \({d}_{j}\left(k\right)\) converge to optimal value. Additionally, \({e}^{-\frac{x}{a}({q}_{i}+a\left(1-{q}_{i}\right))}\) represents the expectation of \({\theta }_{i}(x,q)\) when the reward is applied to the minimum stochastic multi-layer shortest path \({\omega }_{i}\).

$$U{\theta }_{i}\left(x,q\right)=[{q}_{i}{d}_{i}^{*}{e}^{-\frac{x}{a}\left({q}_{i}+a\left(1-{q}_{i}\right)\right)}+\sum_{j\ne i}{q}_{i}{d}_{i}^{*}{e}^{-\frac{x}{a}\left({q}_{i}\left(1-{q}_{i}\right)\prod_{\begin{array}{c}{v}_{m}\notin {\omega }_{j},n=0\\ or\\ {v}_{m}\in {\omega }_{j},n=1\end{array}}\frac{({p}_{m}^{n}+a\left(1-{p}_{m}^{n}\right))}{({p}_{m}^{n}\left(1-a\right))}\right)}]$$

(50)

$$ = \left[{\mathop \sum \limits_{j = i} q_{i} d_{i}^{*} e^{{ - \frac{x}{a}\rho_{i}^{j} \left({q_{i} + a\left({1 - q_{i} } \right)} \right)}} + \mathop \sum \limits_{j \ne i} q_{i} d_{i}^{*} e^{{ - \frac{x}{a}\left({\rho_{i}^{j} q_{i} \left({1 - q_{i} } \right)} \right)}} } \right. $$

(51)

where \({\rho }_{i}^{j}>0\) and is defined as

$$ \rho_{i}^{j} = \left\{ {\begin{array}{*{20}c} {\mathop \prod \limits_{{\begin{array}{*{20}c} {v_{m} \notin \omega_{j} ,n = 0} \\ {or} \\ {v_{m} \in \omega_{j} ,n = 1} \\ \end{array} }} \frac{{\left({p_{m}^{n} + a\left({1 - p_{m}^{n} } \right)} \right)}}{{\left({p_{m}^{n} \left({1 - a} \right)} \right)}}} & {i \ne j} \\ 1 & {i = j \,\,or\,\, \left({\omega_{i} \cap \omega_{j} } \right) = \phi } \\ \end{array} } \right. $$

(52)

$$ U\theta_{i} \left({x,q} \right) - \theta_{i} \left({x,q} \right) = \left[{e^{{ - \frac{{xq_{i} \rho_{i}^{j} }}{a}}} \mathop \sum \limits_{j = i} q_{j} d_{j}^{*} e^{{ - x\left({1 - q_{i} } \right)\rho_{i}^{j} }} + e^{{ - \frac{{xq_{i} \rho_{i}^{j} }}{a}}} \mathop \sum \limits_{j \ne i} q_{j} d_{j}^{*} e^{{xq_{i} \rho_{i}^{j} }} } \right] - e^{{ - \frac{{xq_{i} }}{a}}} $$

(53)

\({\theta }_{i}\left(x,q\right)\) is super-regular if

$$ e^{{ - \frac{{xq_{i} \rho_{i}^{j} }}{a}}} \mathop \sum \limits_{j = i} q_{j} d_{j}^{*} e^{{ - x\left({1 - q_{i} } \right)\rho_{i}^{j} }} + e^{{ - \frac{{xq_{i} \rho_{i}^{j} }}{a}}} \mathop \sum \limits_{j \ne i} q_{j} d_{j}^{*} e^{{xq_{i} \rho_{i}^{j} }} \le e^{{ - \frac{{xq_{i} }}{a}}} $$

(54)

And

$$ U\theta_{i} \left({x,q} \right) \le e^{{ - \frac{{xq_{i} }}{a}}} q_{i} d_{i}^{*} e^{{ - x\left({1 - q_{i} } \right)}} + e^{{ - \frac{{xq_{i} }}{a}}} \mathop \sum \limits_{j \ne i} q_{j} d_{j}^{*} e^{{xq_{i} }} $$

(55)

If \(\theta_{i} \left({x,q} \right)\) is super-regular, so we have

$$ U\theta_{i} \left({x,q} \right) - \theta_{i} \left({x,q} \right) \le \left[{e^{{ - \frac{{xq_{i} }}{a}}} q_{i} d_{i}^{*} e^{{ - x\left({1 - q_{i} } \right)}} + e^{{ - \frac{{xq_{i} }}{a}}} \mathop \sum \limits_{j \ne i} q_{j} d_{j}^{*} e^{{xq_{i} }} } \right] - e^{{ - \frac{{xq_{j} }}{a}}} $$

(56)

After performing some multiplication and division on the right side of the above inequality with \(-x{q}_{i}\), and simplifying the algebraic expressions, we obtain.

$$ \begin{aligned} U\theta_{i} \left({x,q} \right) - & \theta_{i} \left({x,q} \right) \le - xq_{i} e^{{\frac{{ - xq_{i} }}{a}}} \left[{q_{i} d_{i}^{*} \frac{{e^{{ - x\left({1 - q_{i} } \right)}} - 1}}{{ - xq_{i} }} - \mathop \sum \limits_{j \ne i} q_{i} d_{i}^{*} \frac{{e^{{xq_{i} }} - 1}}{{xq_{i} }} \right] \\ & = - xq_{i} e^{{\frac{{ - xq_{i} }}{a}}} \left[{d_{i}^{*} \frac{{e^{{ - x\left({1 - q_{i} } \right)}} - 1}}{ - x} - \mathop \sum \limits_{j \ne i} q_{i} d_{i}^{*} \frac{{e^{{xq_{i} }} - 1}}{{xq_{i} }} \right] \\ & = - xq_{i} e^{{\frac{{ - xq_{i} }}{a}}} \left[{\left({1 - q_{i} } \right)d_{i}^{*} \frac{{e^{{ - x\left({1 - q_{i} } \right)}} - 1}}{{ - x\left({1 - q_{i} } \right)}} - \mathop \sum \limits_{j \ne i} q_{i} d_{i}^{*} \frac{{e^{{xq_{i} }} - 1}}{{xq_{i} }} \right] \\ \end{aligned} $$

(57)

And

$$ V\left[u \right] = \left\{ {\begin{array}{*{20}c} {\frac{{e^{u} - 1}}{u}} & {u \ne 0} \\ 1 & {u = 0} \\ \end{array} } \right. $$

(58)

$$ \theta_{i} \left({x,q} \right) - \theta_{i} \left({x,q} \right) \le - xq_{i} e^{{\frac{{ - xq_{i} }}{a}}} \left[{(1 - q_{i})d_{i}^{*} V\left[{ - x\left({1 - q_{i} } \right)} \right] - \left(\mathop \sum \limits_{j \ne i} q_{i} d_{j}^{*}\right)V\left[{xq_{i} } \right]} \right] = xq_{i} \theta_{i} \left({x,q} \right)G_{i} \left({x,q} \right) $$

(59)

where \(G_{i} \left({x,q} \right)\) is defined as

$$ G_{i} \left({x,q} \right) = \left({1 - q_{i} } \right)d_{i}^{*} V\left[{ - x\left({1 - q_{i} } \right)} \right] - \left({\mathop \sum \limits_{j \ne i} q_{j} d_{j}^{*} } \right)V\left[{xq_{i} } \right] $$

(60)

Therefore, \(\theta_{i} \left({x,q} \right)\) is super-regular if

$$ G_{i} \left({x,q} \right) \ge 0 $$

(61)

For all \(q \in S_{r}\), from Eq. (16), it follows that \(\theta_{i} \left({x,q} \right)\) is super regular if we have

$$ f_{i} \left({x,q} \right) = \frac{{V\left[{ - x\left({1 - q_{i} } \right)} \right]}}{{V\left[{xq_{i} } \right]}} \le \frac{{\mathop \sum \nolimits_{j \ne i} q_{j} d_{j}^{*} }}{{\left({1 - q_{i} } \right)d_{i}^{*} }} $$

(62)

The right side of the inequality (61) contains only non-negative terms, therefore

$$ \left({\mathop \sum \limits_{j \ne i} q_{i} } \right)\begin{array}{*{20}c} {\min } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*} }}{{d_{i}^{*} }}} \right) \le \frac{1}{{\left({1 - q_{i} } \right)}}\mathop \sum \limits_{j \ne i} q_{j} \frac{{d_{j}^{*} }}{{d_{i}^{*} }} \le \left({\mathop \sum \limits_{j \ne i} q_{i} } \right)\begin{array}{*{20}c} {\max } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*} }}{{d_{i}^{*} }}} \right) $$

(63)

By replacing \(\mathop \sum \limits_{j \ne i} q_{j}\) with (1-\(q_{i})\) in the previous inequality, we can rephrase it as

$$ \begin{array}{*{20}c} {\min } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*} }}{{d_{i}^{*} }}} \right) \le \mathop \sum \limits_{j \ne i} q_{j} \frac{{d_{j}^{*} }}{{d_{i}^{*} }} \le \begin{array}{*{20}c} {\max } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*}

$$ }}{{d_{i}^{*} }}} \right) $$

(64)

According to Eq. (18), we can deduce that $\theta_{i} \left({x,q} \right)$ is super-regular if

$$ f_{i} \left({x,q} \right) \ge { }\begin{array}{*{20}c} {\max } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*} }}{{d_{i}^{*} }}} \right)\left({{\text{x}},{\text{q}}} \right) $$

(65)

For further simplification, let's employ logarithms. Let

$$ \Delta \left({q,x} \right) = {\text{Ln}} f_{i} \left({x,q} \right) $$

(66)

It has also been shown in [66, 67] that

$$ - \mathop \int \limits_{0}^{x} H^{\prime}\left(u \right)du \le \Delta \left({q,x} \right) \le - \mathop \int \limits_{{ - x}}^{0} H^{\prime}\left(u \right)du $$

(67)

where $\left(u \right) = \frac{{{\text{dH}}\left(u \right)}}{{{\text{du}}}}$, $H\left(u \right) = \ln V\left(u \right)$ We have

$$ \frac{1}{V\left[x \right]} \le \frac{{V\left[{ - x\left({1 - q_{i} } \right)} \right]}}{{V\left[{xq_{i} } \right]}} \le V\left[{ - x} \right] $$

(68)

And

$$ \frac{1}{V\left[x \right]} = \begin{array}{*{20}c} {\max } \\ {j \ne i} \\ \end{array} \left({\frac{{d_{j}^{*} }}{{d_{i}^{*} }}} \right) $$

(69)

Let's denote the value of \({\text{x}}\) that satisfies Eq. (68) as \({\text{x}}^{*}\). It has been demonstrated that there exists a positive \(x\) value that satisfies Eq. (68) when \(\left({\frac{{d_{j} }}{{d_{i} }}} \right)\) is less than 1 for all \(j \ne i\). By selecting \(x = x^{*}\), Eq. (68) remains true. As a result, Eq. (60) is also true, and \(\theta_{i} \left({x,q} \right)\) is a super regular function. Thus,

$$ \varphi_{i} \left[{x,q} \right] = \frac{{e^{{\frac{{ - xq_{i} }}{a}}} - 1}}{{e^{{\frac{ - x}{a}}} - 1}} $$

(70)

Considering a function that meets the boundary conditions (40) and is sub-regular, along with the information from theorem 2 and inequality, we can reach the following conclusion:

$$ \varphi_{i} \left[{x,q} \right] \le {\Gamma }_{i} \left(q \right) \le 1 $$

(71)

By observing the definition of \(\varphi_{i} \left[{x,q} \right]\), we can notice that for any given \({\upvarepsilon } > 0\), there exists a positive constant \({\text{a}}^{*} < 1\). This means that for all \(0 < {\text{a}} \le {\text{a}}^{*}\) the following condition is satisfied

$$ 1 - \varepsilon \le \phi_{i} \left[{x,q} \right] \le {\Gamma }_{i} \left(q \right) \le 1 $$

(72)

So, the conclusion is that as \(k\) (the number of iterations) approaches infinity, the probability of Algorithm 1 converging to SMSP with the minimum expected weight is equal to 1.

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